



Definition of the feasible solution set in multicriteria optimization problems with continuous, discrete, and mixed design variables

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ABSTRACT

Engineering optimization problems are multicriteria with continuous, discrete, and mixed design variables. Correct definition of the feasible solution set is of fundamental importance in these problems. It is quite difficult for the expert to define this set. For this reason, the results of searching for optimal solutions frequently have no practical meaning. Furthermore, correct definition of this set makes it possible to significantly reduce the time of searching for optimal solutions. This paper describes construction of the feasible solution set with continuous, discrete, and mixed design variables on the basis of Parameter Space Investigation (PSI) method.

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1. Introduction

Existing optimization methods tacitly assume that the expert can state an engineering optimization problem correctly, in particular to define the feasible solution set. As a rule, definition of the feasible solution set represents significant difficulties and existing optimization methods are not helpful in this situation. Thus, in the overwhelming majority of cases, the expert ends up solving ill-posed problems [1–4].

Prior research considered approaches for constructing the feasible solution set and searching for optimal solutions in multicriteria problems with continuous design variables [1–4]. Although there exist approximate methods for searching for optimal solutions in problems with discrete design variables (e.g., genetic and stochastic approximation algorithms [5–11]), approaches for constructing the feasible solution set in such problems have not been discussed previously. Similarly, the techniques for constructing the feasible solution set in problems with mixed (e.g., combination of continuous and discrete) design variables have not received attention in prior literature. The present paper presents a universal mechanism for generating the feasible solution set for problems with continuous, discrete, and mixed design variables on the basis of Parameter Space Investigation (PSI) method [1–4].

In this work we consider construction of the feasible solution set for a class of engineering optimization problems that possess the following distinctive features:

- I. The problems are essentially multicriteria, and the criteria are usually contradictory. For this reason there are difficulties in defining the criteria constraints correctly.
- II. The initial constraints on the design variables, criteria, and functional dependencies may result in an empty or very sparse feasible solution set.

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- III. The feasible solution set can be multiply connected, and its volume may be several orders of magnitude smaller than that of the domain where the optimal solutions are sought. Generally, the feasible solution set is non-convex.
- IV. Mathematical models are often complex systems of equations (including differential equations) that may be nonlinear, deterministic or/and stochastic, with distributed, lumped, continuous, discrete, and mixed design variables. Information about the smoothness of goal functions is usually not available.
- V. In this class of problems, experts do not encounter difficulties in analyzing the Pareto optimal set and choosing the most preferred solution. This is because experts have a sufficiently well-defined system of preferences¹ and the Pareto optimal set often contains a small number of solutions due to stringent constraints.

The remainder of this paper is organized as follows: In Sections 2–4 we provide definition of the feasible solution set and approaches for its construction. Tools for visualization that are instrumental for construction of the feasible solution set are described in Section 5. Examples with continuous, discrete, mixed design variables are described in Sections 6 and 7. The paper concludes with Section 8.

2. Formulation of multicriteria optimization problem

We assume that an object depends on r design variables $\alpha_1, \dots, \alpha_r$ representing a point $\alpha = (\alpha_1, \dots, \alpha_r)$ in the r -dimensional space. In the general case, one has to take into account design variable, functional, and criteria constraints. The *design variable constraints* have the form $\alpha_j^* \leq \alpha_j \leq \alpha_j^{**}, j = 1, \dots, r$. Constraints α_j^* and α_j^{**} define a parallelepiped Π in the r -dimensional design variable space. The *functional constraints* can be written as follows: $C_l^* \leq f_l(\alpha) \leq C_l^{**}, l = 1, \dots, t$, where $f_l(\alpha)$ is a functional relation, C_l^* and C_l^{**} are some constants. The operation of an object is described by the particular performance criteria $\Phi_v(\alpha), v = 1, \dots, k$. All other things being equal, it is desired that these criteria are optimized. For simplicity, we assume that functions $\Phi_v(\alpha)$ are to be minimized. To avoid situations in which the expert regards the values of some criteria as unacceptable, we introduce *criterion constraints* in the form $\Phi_v(\alpha) \leq \Phi_v^{**}, v = 1, \dots, k$, where Φ_v^{**} is the worst value of criterion $\Phi_v(\alpha)$ acceptable to an expert. The choice of Φ_v^{**} is discussed in the following section. The design variable, functional, and criteria constraints define the feasible solution set $D \subset \Pi$.

3. Construction of feasible solution set on the basis of Parameter Space Investigation (PSI) method

We propose to construct the feasible solution set using the PSI method that is based on the investigation of parallelepiped Π with α^i points of uniformly distributed sequences. The method is extensively described in [1–4], and here we present only its summary. The PSI method consists of three stages:

Stage 1: Compilation of test tables via computer. First, one chooses N trial points $\alpha^1, \dots, \alpha^N$ that satisfy the functional constraints. Then all the particular criteria $\Phi_v(\alpha^i)$ are calculated at each of the points α^i ; and for each of the criteria a test table is compiled so that the values of $\Phi_v(\alpha^1), \dots, \Phi_v(\alpha^N)$ are arranged in increasing order, i.e.

$$\Phi_v(\alpha^{i_1}) \leq \Phi_v(\alpha^{i_2}) \leq \dots \leq \Phi_v(\alpha^{i_N}), \quad v = 1, \dots, k, \quad (1)$$

where i_1, i_2, \dots, i_N are the numbers of trials (a separate set for each v). Taken together, the k tables form a complete test table.

Stage 2: Preliminary selection of criterion constraints. This stage includes interaction with an expert. By analyzing inequalities (1), an expert specifies the criteria constraints Φ_v^{**} . An expert analyzes one test table and imposes the criterion constraint. Then one proceeds to the next table, and so on. Note that the revision of the criteria constraints within the limit of the test tables does not lead to any difficulties for an expert.

Since we want to minimize all criteria, Φ_v^{**} are the maximum values of the criteria $\Phi_v(\alpha)$, which guarantee an acceptable level of the object's operation. If the selected values of Φ_v^{**} are not a maximum, then many interesting solutions may be lost, since some of the criteria are contradictory. As a rule, an expert may set Φ_v^{**} equal to a criterion value $\Phi_v(\bar{\alpha})$ whose feasibility is beyond doubt. However, if one starts by determining the maximum possible value of Φ_v^{**} then one has to proceed to Stage 3.

Stage 3: Verification that the set D is non-empty. Let us fix a criterion, say $\Phi_{v_1}(\alpha)$, and consider the corresponding test table (2). Let S_1 be the number of the values in the table satisfying the selected criterion constraint:

$$\Phi_{v_1}(\alpha^{i_1}) \leq \dots \leq \Phi_{v_1}(\alpha^{i_{S_1}}) \leq \Phi_{v_1}^{**} = \Phi_{v_1}(\bar{\alpha}). \quad (2)$$

One should choose the criterion Φ_{v_1} for which S_1 is the minimum among the analogous numbers calculated for each of the criteria Φ_v . Then criterion Φ_{v_2} is selected by analogy with Φ_{v_1} and the values of $\Phi_{v_2}(\alpha^{i_1}), \dots, \Phi_{v_2}(\alpha^{i_{S_1}})$ in the test table are considered. Let the table contain $S_2 \leq S_1$ values such that $\Phi_{v_2}(\alpha^{i_j}) \leq \Phi_{v_2}^{**}$, where $1 \leq j \leq S_2$. Similar procedures are carried out for each criterion. Then if at least one point can be found for which all criteria constraints are valid simultaneously, then

¹ More complex cases of the decision making, where preferences are not necessarily stable on the Pareto optimal set are discussed in [16].

the set D is non-empty. However, if the concessions are highly undesirable, then one may return to Stage 1 and increase the number of points in order to repeat Stage 2 and Stage 3 using extended test tables.

The majority of constraints on functional dependencies C_j^* and C_j^{**} are “soft”, i.e. they can be changed. If functional constraints are poorly specified, this can considerably reduce the feasible solution set. Therefore, many interesting solutions become unreasonably unfeasible. Furthermore, the feasible solution set can be empty. For this occasion, we represent the functional dependencies with “soft” constraints in the form of so-called pseudo-criteria. Thus, criteria vector contains both performance criteria and pseudo-criteria. The PSI method helps to define pseudo-criteria constraints with respect to improving the performance criteria.

The PSI method is implemented in the MOVI (Multicriteria Optimization and Vector Identification) software system [12].

4. Uniformly distributed sequences in the problems with continuous, discrete, and mixed design variables

As mentioned above PSI method uses uniformly distributed sequences to investigate design variable space. At present, the so-called LP_τ sequences are among the best ones in terms of uniformity characteristics. We note that other generators, including the random number generators, are used as well in the PSI method.

Below we describe approaches for generating uniform sequences in the problems with continuous, discrete, and mixed design variables.

Continuous design variables: If points Q_i $i = 1, \dots, N$ with Cartesian coordinates (q_{i1}, \dots, q_{ir}) form a uniformly distributed sequence in unit cube K^r , then points α^i , with coordinates $\alpha_1^i, \dots, \alpha_r^i$, where

$$\alpha_j^i = \alpha_j^* + q_{ij}(\alpha_j^{**} - \alpha_j^*), \quad j = 1, 2, \dots, r,$$

form a uniformly distributed sequence in parallelepiped π consisting of points $(\alpha_1, \dots, \alpha_r)$ whose coordinates satisfy inequalities $\alpha_j^* \leq \alpha_j \leq \alpha_j^{**}$ [1–4]. Points Q_i can be independent random numbers as well.

Discrete design variables: First we introduce an extension of the definition of objective function that substantiates application of the PSI method in problems with discrete design variables.

Consider the objective function $f(x, z)$, where point x belongs to an n -dimensional parallelepiped Π , and the design variable z is discrete and may take m values: $\{\hat{z}_1, \dots, \hat{z}_m\}$. We extend the definition of $f(x, z)$ to all possible z from the interval $0 \leq z < 1$. Let $k = 1, 2, \dots, m$. If

$$\frac{k-1}{m} \leq z < \frac{k}{m}, \quad (3)$$

then we will consider that

$$f(x, z) = f(x, \hat{z}_k). \quad (4)$$

Thus, the objective function is defined in the $(n+1)$ -dimensional parallelepiped $\Pi \times [0, 1)$. For the value of z we choose the $(n+1)$ th coordinate of a quasi-random point $(q_1, q_2, \dots, q_{n+1})$ (or an independent random number). According to [15], the value of k from (3) may be defined by the formula $k = 1 + [mq_{n+1}]$. This value of k should be used in (4). It is evident that the frequency of appearance of \hat{z}_k in (4) is equal to $1/m$ for each k .

Mixed design variables: Mixed design variables are $\bar{\alpha}^i = (\alpha_j^i, \hat{z}_k^i)$, where $j = 1, \dots, n$ indices continuous and $k = n+1, \dots, r$ indices discrete design variables. The treatment of continuous and discrete design variables is analogous to the above description.

5. Tools for visualization

Below we describe a few visualization tools that are particularly useful for construction of the feasible solution set [13]. It is important to emphasize that these tools should be used together with the test tables. All the tools² listed below are implemented in the software system MOVI.

- Histograms of the distribution of feasible solutions. The intervals $[\alpha_j^*, \alpha_j^{**}]$, $j = 1, \dots, r$ are divided into ten identical subintervals. Above each subinterval, the number of feasible designs entering this subinterval is indicated. Analyzing the histograms reveals how the feasible solution set is distributed in design variable space. The histograms play the main role in correcting design variable and other constraints.
- Graphs “Criterion vs. Design Variable.” We consider projections of the *multidimensional points* $\Phi_v(\alpha^i)$, $v = 1, \dots, k$, $i = 1, \dots, N1$ onto the plane $\Phi_v \alpha_j$. These projections provide information about the sensitivity of criteria to the design variables, and also point to localization of the feasible solution set. Significance of the sensitivity of criteria to the design variables is indicated in [14].
- Graphs “Criterion vs. Criterion”. After N tests, $N1$ design variable vectors have entered the test table. We consider projections of the *multidimensional points* $\Phi_v(\alpha^i)$, $v = 1, \dots, k$, $i = 1, \dots, N1$ onto the plane $\Phi_i \Phi_j$. These projections provide information about dependencies between criteria and localization of the feasible solution set in criteria space.

The examples of application of the above tools are provided in the next section.

² In this paper we describe only a few basic tools. A detailed description of other tools, e.g. tables of functional and criteria failures, tables of criteria, is provided in [13].

Table 1

Inequalities (6), (6a) and (6b) that define design variable constraints.

Continuous design variables (6): Example I	Discrete design variables (6a): Example II	Mixed design variables (6b): Example III
$1.1 \times 10^6 \leq \alpha_1 \leq 2.0 \times 10^6$	$\alpha_1 = \{1.1 \times 10^6; 1.3 \times 10^6; 1.5 \times 10^6; 1.7 \times 10^6; 1.8 \times 10^6; 2.0 \times 10^6\}$	$\alpha_1 = \{1.1 \times 10^6; 1.3 \times 10^6; 1.5 \times 10^6; 1.7 \times 10^6; 1.8 \times 10^6; 2.0 \times 10^6\}$
$4.0 \times 10^4 \leq \alpha_2 \leq 5.0 \times 10^4$	$\alpha_2 = \{4.0 \times 10^4; 4.3 \times 10^4; 4.7 \times 10^4; 4.8 \times 10^4; 4.9 \times 10^4; 5.0 \times 10^4\}$	$\alpha_2 = \{4.0 \times 10^4; 4.3 \times 10^4; 4.7 \times 10^4; 4.8 \times 10^4; 4.9 \times 10^4; 5.0 \times 10^4\}$
$950 \leq \alpha_3 \leq 1050$	$\alpha_3 = \{950; 970; 980; 990; 1000; 1040; 1050\}$	$950 \leq \alpha_3 \leq 1050$
$30 \leq \alpha_4 \leq 70$	$\alpha_4 = \{30; 32; 37; 40; 45; 50; 60; 70\}$	$30 \leq \alpha_4 \leq 70$
$80 \leq \alpha_5 \leq 120$	$\alpha_5 = \{80; 85; 90; 95; 100; 105; 120\}$	$80 \leq \alpha_5 \leq 120$

6. Definition of the feasible solution set

This section shows the process of definition of the feasible solution set by an example. Consider the motion of system governed by the following equations:

$$\begin{aligned} M_1 X_1'' + C(X_1' - X_2') + K_1 X_1 + K_2(X_1 - X_2) &= P \cdot \cos(\omega t) \\ M_2 X_2'' + C(X_2' - X_1') + K_2(X_2 - X_1) &= 0. \end{aligned} \quad (5)$$

The mass M_1 is attached to a fixed base by a spring with stiffness coefficient K_1 . A spring-and-dashpot element with stiffness coefficient K_2 and damping coefficient C is located between masses M_1 and M_2 . The harmonic force acts upon mass M_1 . The amplitude and frequency of the exciting force are identified as $P = 2000$ (N) and $\omega = 30$ (s^{-1}). We treat parameters K_1 , K_2 , M_1 , M_2 , and C as the design variables to be determined, i.e.: $\alpha_1 = K_1$ (N/m); $\alpha_2 = K_2$ (N/m); $\alpha_3 = M_1$ (kg); $\alpha_4 = M_2$ (kg); $\alpha_5 = C$ (N s/m). The design variable constraints are defined as the parallelepiped Π defined by the inequalities (6), (6a) and (6b) shown in Table 1. There are three functional dependencies with five constraints (on the total mass and on the partial frequencies):

$$\begin{aligned} f_1(\alpha) &= \alpha_3 + \alpha_4 \leq 1100.0 \text{ kg}; \\ 33.0 \leq f_2(\alpha) &= p_1 = \sqrt{\alpha_1/\alpha_3} \leq 42.0 \text{ (s}^{-1}\text{)}; \\ 27.0 \leq f_3(\alpha) &= p_2 = \sqrt{\alpha_2/\alpha_4} \leq 32.0 \text{ (s}^{-1}\text{)}. \end{aligned} \quad (7)$$

An expert provides the following recommendations: Values of design variable and functional constraints can be changed, if that leads to the improving the main criteria.

The upper limits on the functions f_2 and f_3 are defined approximately and can be significantly modified. In other words, these are two “soft” functional constraints $f_2(\alpha) \leq 42.0$, $f_3(\alpha) \leq 32.0$. The rest of functional constraints $f_1(\alpha) \leq 1100.0$; $33.0 \leq f_2(\alpha)$; $27.0 \leq f_3(\alpha)$ are rigid.

Taking into account these considerations, we formulate the optimization problem as follows.

Criteria: According to the PSI method, in order to define “soft” constraints on functional relations f_2 and f_3 , the latter should be interpreted as the *pseudo-criteria*, i.e. $\Phi_1 = f_2$ and $\Phi_2 = f_3$. The system is to be minimized with respect to the following four performance criteria:

- $\Phi_3 = X_{1\theta}$ mm—vibration amplitude of the first mass;
- $\Phi_4 = M_1 + M_2$ kg—metal consumption of the system;
- $\Phi_5 = X_{1\theta}/X_{1st}$ —dimensionless dynamical characteristic of the system, where X_{1st} is the static displacement of mass M_1 under the action of the force P ;
- $\Phi_6 = \omega/p_1$ —dimensionless dynamical characteristic of the system.

Thus, we have a vector of criteria $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6)$, on the basis of which test tables are constructed and criteria constraints are defined.

Design variables: There are five design variables: $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ that can either be continuous (Example I), discrete (Example II), or mixed (the first and second design variables are discrete; third, fourth, and fifth are continuous) (Example III) as described in Table 1.

Functional constraints: As mentioned above, we have three rigid functional constraints.

Criteria constraints: We define criteria constraints on the basis of the PSI method:

$$\Phi_1^{**} = 35.20; \quad \Phi_2^{**} = 36.98; \quad \Phi_3^{**} = 8.40; \quad \Phi_4^{**} = 1019; \quad \Phi_5^{**} = 18.11, \quad \text{and} \quad \Phi_6^{**} = 0.9. \quad (8)$$

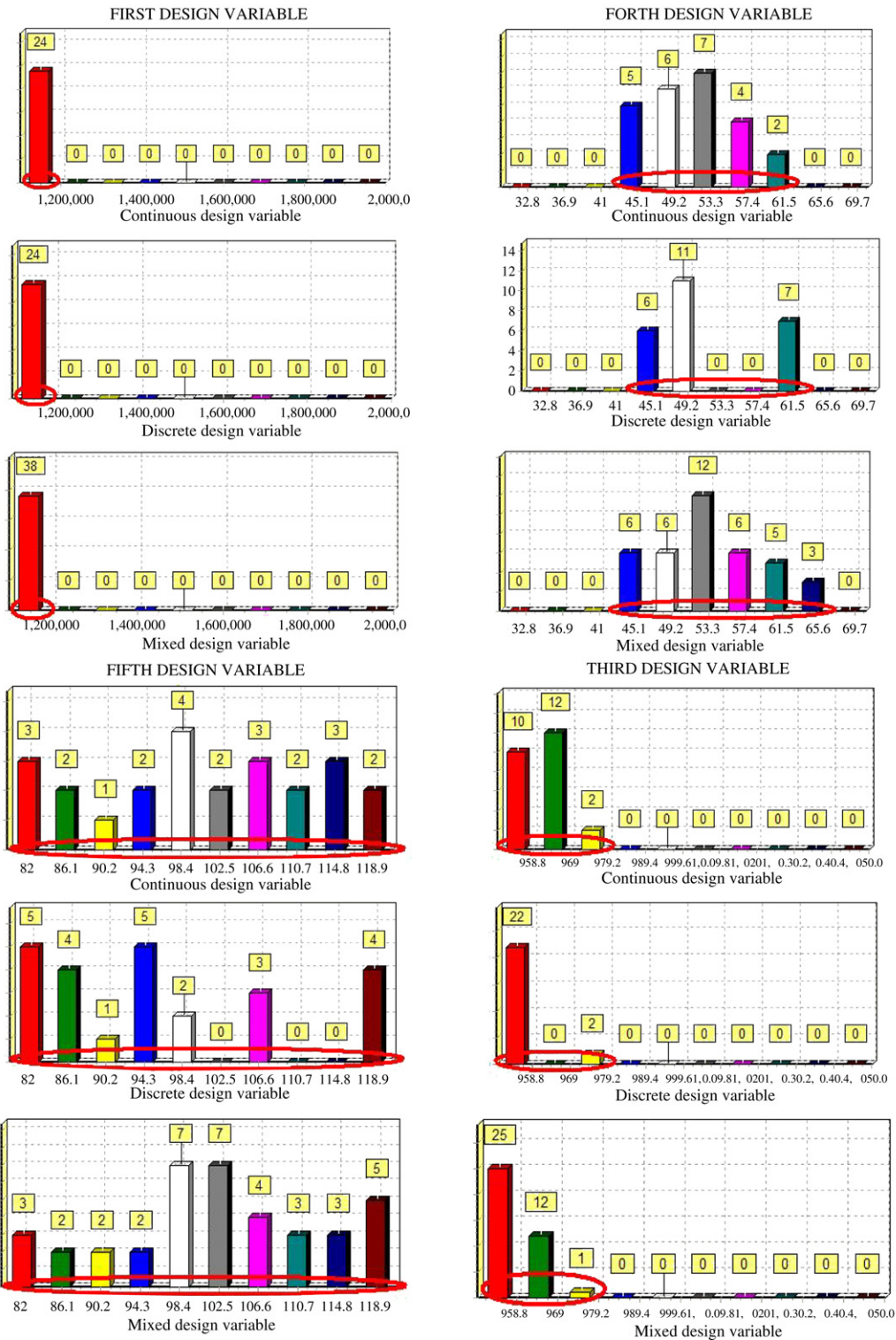


Fig. 1. Histograms.

Notice that constraints (6), (7) and (8) define the feasible solution set for Example I; constraints (6a), (7) and (8) define the feasible solution set of Example II; and constraints (6b), (7) and (8) define the feasible solution set of Example III.

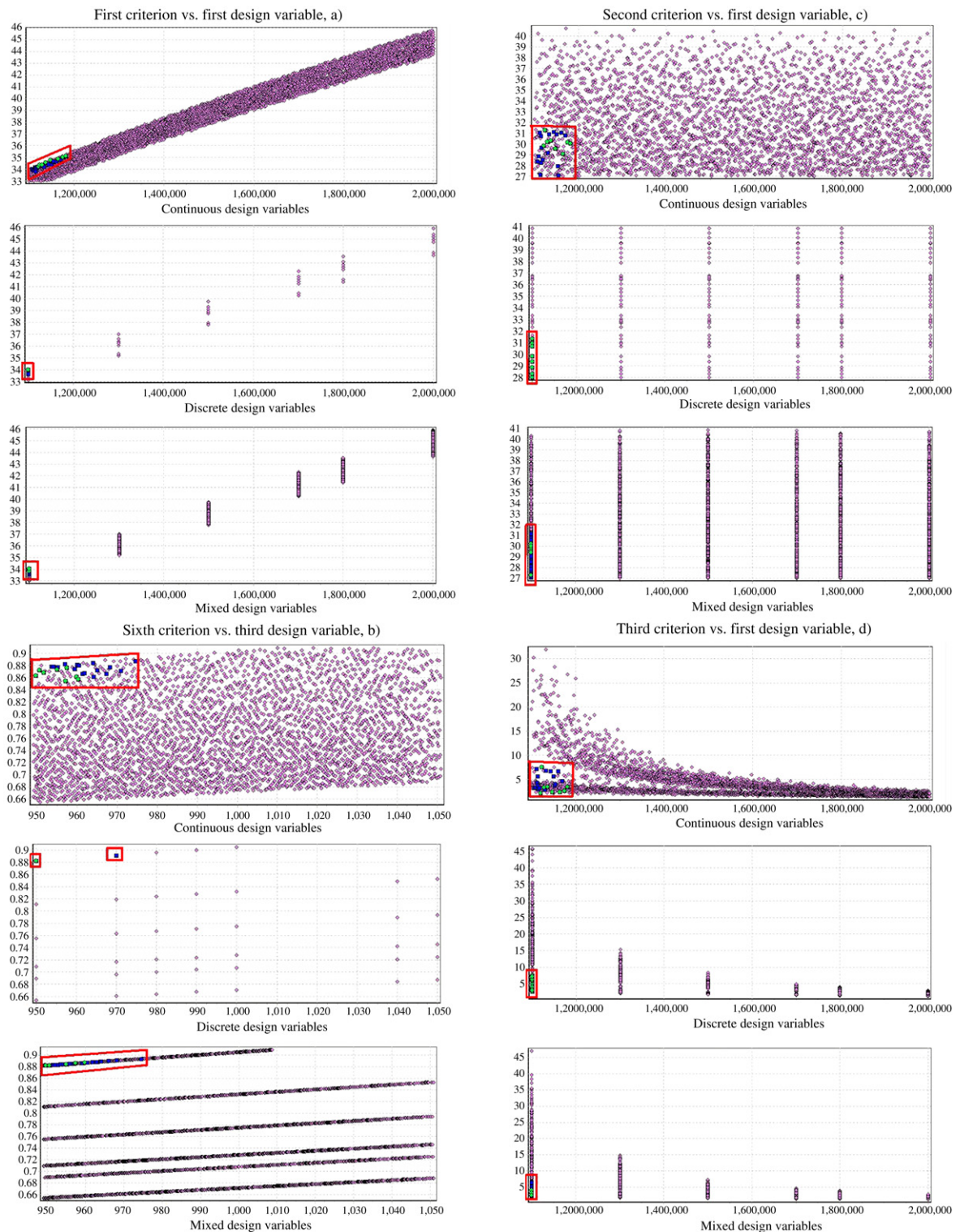


Fig. 2. Graphs "Criterion vs. Design Variable".

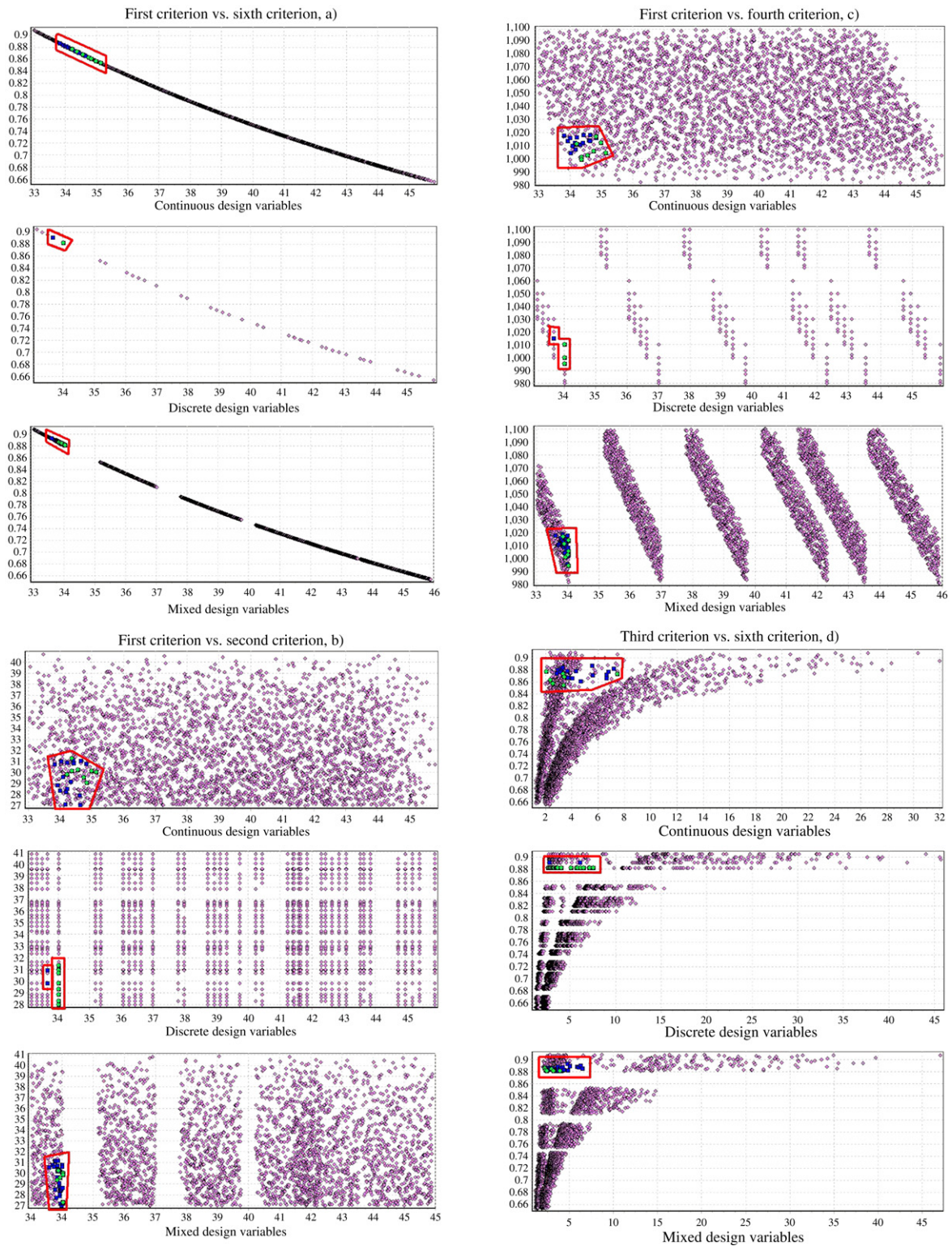


Fig. 3. Graphs "Criterion vs. Criterion"

Table 2

Location of the feasible solutions.

Problems	First design variable	Second design variable	Third design variable	Fourth design variable	Fifth design variable
Continuous design variables: Example I	$[1.1 \times 10^6; 1.18 \times 10^6]$	$[4.0 \times 10^4; 5.0 \times 10^4]$	[950; 975]	[42.2; 61]	[81.4; 120]
Discrete design variables: Example II	1.1×10^6	$4.0 \times 10^4; 5.0 \times 10^4$	950; 970	45; 60	80; 120
Mixed design variables: Example III	1.1×10^6	$4.0 \times 10^4; 5.0 \times 10^4$	[950; 975]	[42.2; 65.7]	[82.9; 120]

7. Results

We performed $N = 4096$ tests in each of the three examples and obtained 24, 24, and 38 feasible solutions, respectively. The location feasible solution is shown in Table 2, histograms (Fig. 1), graphs “Criterion vs. Design Variable” (Fig. 2),³ and graphs “Criterion vs. Criterion” (Fig. 3).⁴ In the above figures, the area with feasible solutions is shown with red circle or polygon; the unfeasible solutions are shown with magenta points in Figs. 2 and 3. Similar results were obtained for $N = 2048$ and $N = 8192$ tests (not shown here for brevity).

In cases with continuous, discrete, and mixed design variables, there is the same tendency of location of the feasible solution set, regardless of the types of design variables. From the histograms in Fig. 1 it follows that localization of feasible solutions occurs at the left-hand limits for the first and third design variables, at the midpoint of the interval for the fourth design variable, while for the fifth design variable, feasible solutions are distributed throughout the entire interval. The nature of the dependencies of criteria on design variables (Fig. 2) and of criteria on criteria (Fig. 3) is nearly identical for all three cases. The location of the feasible solutions on these graphs is also very similar.

From the example with continuous design variables (Example I), it follows that the volume of the feasible solution set is ~ 40 times smaller than the volume of the initial parallelepiped; see (6) and Table 2. The initial statement of the problem, including limits of the design variables, is usually corrected on the basis of analysis of the resulting feasible solution set. This in turn almost always leads to an improvement in obtained results [13]. Specifically, for the first, third, and fourth design variables, this can mean a revision of the initial limits (6) and generation of new limits, for example, $1.0 \times 10^6 \leq \alpha_1 \leq 1.18 \times 10^6$, $950 \leq \alpha_3 \leq 975$, and $42.2 \leq \alpha_4 \leq 61$; see Table 2 and Fig. 1. The same also holds for problems with discrete and mixed design variables (i.e., Examples II and III).

Analysis of criteria vs. design variable and criteria vs. criteria dependencies (Figs. 2 and 3) provides valuable information on the sensitivity of criteria to design variables and on the relationships between criteria. For example, from analysis of graphs in Fig. 2 it follows that there is a linear dependence of the first criterion on the first design variable, there is a fairly complex relation between the second criterion and the first design variable, and so on. From Fig. 3 it follows that the first and sixth criteria are linearly dependent but antagonistic, there is a complex relation between the first and second criteria, and so on. This analysis, like an analysis of the histograms, facilitates correction of the initial statement of the problem.

8. Conclusion

Due to the difficulties in defining the feasible solution set, it is often impossible in the engineering problems to perform a search for optimal solutions or it is often performed ineffectively. In a number of cases, the search for optimal solutions requires an unjustifiably large computing experiment, and the results of the optimization have no applied significance. The PSI method has proved its effectiveness in constructing the feasible solution set in multicriteria problems with continuous, discrete, and mixed design variables. The results presented here show the expediency of using the PSI method for defining the feasible solution set regardless of the nature of the design variables.

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³ In Fig. 2, the values of parameters (design variables) and criteria are shown on the horizontal and vertical axes, respectively.

⁴ In Fig. 3(a), (b), and (c) the value of criterion 1 is shown on the horizontal axis; in Fig. 3(d) the value of criterion 3 is shown on the horizontal axis; values of the rest of the criteria are shown on the vertical axis.

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