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Management of constraints in optimization problems

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ABSTRACT

Optimization problems are encountered in all scientific disciplines and in many aspects of everyday life. Application of established optimization methods assumes that an expert can state the optimization problem correctly. Unfortunately, this is not the case in reality. Below we consider how to help an expert state and solve optimization problems. The proposed technique splits constraints of an optimization problem into "soft" (manageable) and "rigid" constraints and modifies the "soft" constraints in an interactive mode with the expert. The technique is illustrated with a numeric example devoted to the optimization of a real-life nonlinear system.

1. Introduction

The basic components of optimization problems include: *design variables* $\alpha_1, \ldots, \alpha_r$ with *design variable constraints* $\alpha_j^* \leq \alpha_j \leq \alpha_j^{**}, j = 1, \ldots, r$; *performance criteria* $\Phi_\nu(\alpha), \nu = 1, \ldots, k$ (goal functions that we seek to minimize) with *criteria constraints* $\Phi_\nu^{**} \leq \Phi_\nu(\alpha)$; and *functional dependencies* $f_l(\alpha), l = 1, \ldots, t$ with *functional constraints* $C_l^* \leq f_l(\alpha) \leq C_l^{**}$ [1–4]. Unlike criteria, functional dependences do not need to be optimized; we only need to satisfy their constraints.

All the above constraints define the *feasible solution set*(D), a region in the criteria and design variable spaces where the optimal solutions should be sought. Determination of the feasible solution set is the essence of a problem statement. An important subset of the feasible solution set contains solutions that cannot be improved by all criteria simultaneously and is referred to as *Pareto optimal solution set.* In order to solve the optimization problem, one has to identify the Pareto optimal set. Obviously, if the feasible solution set is not determined correctly, the resulting Pareto solutions cannot have practical value because they were sought in the wrong place; furthermore many interesting solutions become unreasonably unfeasible.

The larger the number of performance criteria, the greater the amount of information obtained about resources of improving an object and precision of the model used to calculate criteria. Since many criteria are contradictory, definition of criteria constraints represents significant, sometimes insurmountable difficulties. It is worthwhile to notice that numerous attempts to reduce multicriteria problems to single-criterion ones have proved to be fruitless.

We recognize two types of functional constraints: "rigid" and "soft" ("non-rigid"). For example, standards are "rigid" functional constraints. These constraints are not supposed to be changed; they are specified in advance. On the other hand, "soft" functional constraints (e.g. overall dimensions) can be changed, if these revisions lead to the improvement of performance criteria. It is worthwhile to notice that in case of unjustifiably strong constraints on functional dependences, many solutions can become unfeasible.

Design variables are changed within some specified boundaries. Quite often these boundaries can be revised, if it leads to improvement of the values of the main criteria.

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In a traditional statement of optimization problems, constraints are usually given *a priori*. However, it is unlikely that such constraints are correct, especially given high dimensionality of the problems and complexity of underlying models. That is why it is necessary to ensure correctness of any given constraints. Otherwise, the optimization can lead to meaningless results or equally to the loss of important solutions. As was mentioned above, established optimization methods do not address the problem of definition of the feasible solution set.

2. Construction of the feasible solution set

2.1. Parameter Space Investigation (PSI method)

In order to construct the feasible solution set, a method called the Parameter Space Investigation (PSI) has been created and successfully integrated into various fields of industry, science, and technology [1–5]. This method has been used in designing the space shuttle, nuclear reactors, unmanned vehicles, cars, ships, metal-tools, etc. The PSI method performs systematic investigation of the multidimensional domain by using uniformly distributed sequences. Among uniformly distributed sequences known at present, the so-called LP_τ sequences are among the best ones with regards to uniformity characteristics, see [1–4]. Nets or quasi-random points can be also used in the PSI method [5,6]. The PSI method works as follows: *First*, a computer generates design variable vectors. For the vectors that satisfy "rigid" functional constraints, the values of criteria are computed. For each criterion, all the values are arranged in a test table in order (from best to worst values). *Second*, one chooses preliminarily criteria constraints via dialogs with the computer. *Third*, the problem's solvability is verified: if vectors satisfying simultaneously all constraints are determined, the feasible solution set is nonempty, and the problem is solvable. Otherwise, one has to either modify the values of constraints or return to the first step and increase the number of trials. The procedure is iterated until the feasible solution set is nonempty. Finally, the Pareto set is constructed and analyzed.

The empirical success of the PSI method [1–4] can be attributed primarily to the following two reasons: (i) the method allows one to formulate and solve the optimization problem in a single process, and (ii) an expert is often ready to change the constraints given that these changes lead to the improvement of the values of the main criteria.

While solving real-life optimization problems, experts often do not encounter serious difficulties in analyzing the Pareto optimal set and in choosing the most preferred solution. This is because experts have a sufficiently well-defined system of preferences[1] in these types of problems, and the Pareto optimal set often contains a small number of solutions due to stringent constraints.

2.2. "Soft" functional constraints and pseudo-criteria

In the traditional approach to multicriteria problems, one tries to reduce the number of criteria, replacing them with functional dependences with given constraints. From the standpoint of our technique, it is necessary to act on the contrary.

In the case of unjustifiably strong constraints $C_l^*(C_l^{**})$ on functional dependences, many solutions can become unfeasible. For this reason the feasible solution set can be poor or even empty. Therefore it is very important to help the expert determine the "soft" functional constraints correctly.

Let us assume that $f_l(\alpha) \leq C_l^{**}, l = 1, \dots, t$, where C_l^{**} are the "soft" constraints. The concept of pseudo-criteria is presented as following: Instead of the function $f_l(\alpha)$, we introduce an additional criterion $\Phi_{k+l}(\alpha) = f_l(\alpha)$, which we call a pseudo-criterion. To find the value of the constraint Φ_{k+l}^{**} one has to compile a test table containing $\Phi_{k+l}(\alpha)$. By using the PSI method, one can define Φ_{k+l}^{**} in a way that prevents the loss of interesting solutions. In general, when solving the problem with "soft" functional constraints, one has to find the set D, taking all performance criteria and pseudo-criteria into account. In other words, one must solve the problem with the constraints $\Phi_\nu(\alpha) \leq \Phi_\nu^{**}, \nu = 1, \dots, k, k+1, \dots, k+t$. Thus, to define the feasible solution set, we consider a multicriteria problem with $k+t$ criteria. Notice, however, that the pseudo-criteria are not considered when constructing the Pareto optimal set.

It is worthwhile to mention that many single-criterion problems have "soft" functional constraints as well. In these cases, the definition of the feasible solution set is also very important. For determining this set, it is necessary to represent respective dependencies as pseudo-criteria. In other words, we have to consider such single-criterion problems as multicriteria ones. Test tables then will contain one performance criterion and the rest are pseudo-criteria.

3. Example

Below we illustrate by an example some aspects of constructing the feasible set using the PSI method. Consider a statement of an optimization problem and the typical difficulties encountered by an expert at this stage [3]. Two iterations of the statement and the solution of the problem were performed. Each iteration was based on 1024 design variable vectors generated with uniformly distributed sequences.

[1] More complex cases of the decision making, where the preferences are not necessarily stable are discussed in [7].

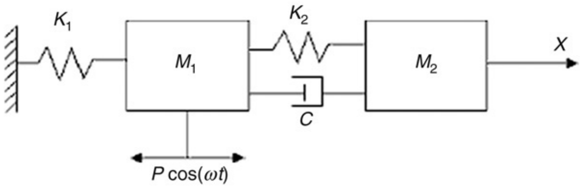

Fig. 1. Vibratory system.

3.1. Description of the vibratory system and initial statement of the optimization problem

The vibratory system consists of two bodies with masses M_1 and M_2, see Fig. 1. The mass M_1 is attached to a fixed base by a spring with stiffness coefficient $\mathcal{K}_1$. A spring-and-dashpot element with stiffness coefficient $\mathcal{K}_2$ and damping coefficient C is located between masses M_1 and M_2. The harmonic force acts upon mass M_1. The amplitude and frequency of the exciting force are identified as $P = 2000$ (N) and $\omega = 30$ (s^{-1}).

The motion of this system is governed by the equations:

$$\begin{aligned} &M_1X_1'' + C(X_1' - X_2') + \mathcal{K}_1X_1 + \mathcal{K}_2(X_1 - X_2) = P \cdot \cos(\omega t) \\ &M_2X_2'' + C(X_2' - X_1') + \mathcal{K}_2(X_2 - X_1) = 0. \end{aligned} \tag{1}$$

We treat the parameters $\mathcal{K}_1, \mathcal{K}_2, M_1, M_2$, and C as the design variables to be determined, i.e.

$$\alpha_1 = \mathcal{K}_1, \qquad \alpha_2 = \mathcal{K}_2, \qquad \alpha_3 = M_1, \qquad \alpha_4 = M_2, \qquad \alpha_5 = C.$$

The design variable constraints are prescribed as the parallelepiped Π defined by the inequalities:

$$\begin{aligned} &1.1 \times 10^6 \le \alpha_1 \le 2.0 \times 10^6 \text{ (N/m)}; \\ &4.0 \times 10^4 \le \alpha_2 \le 5.0 \times 10^4 \text{ (N/m)}; \\ &950 \le \alpha_3 \le 1050 \text{ (kg)}; \\ &30 \le \alpha_4 \le 70 \text{ (kg)}; \\ &80 \le \alpha_5 \le 120 \text{ (N s/m)}. \end{aligned} \tag{2}$$

There are three functional dependencies with five constraints (on the total mass and on the frequencies):

$$\begin{aligned} &f_1(\alpha) = \alpha_3 + \alpha_4 \le 1100.0 \text{ (kg)}; \\ &33.0 \le f_2(\alpha) = p_1 = \sqrt{\alpha_1/\alpha_3} \le 42.0 \left(\text{s}^{-1}\right); \\ &27.0 \le f_3(\alpha) = p_2 = \sqrt{\alpha_2/\alpha_4} \le 32.0 \left(\text{s}^{-1}\right). \end{aligned} \tag{3}$$

The upper limits on the functions f_2 and f_3 are defined approximately and can be significantly modified. In other words, we have two "soft" functional constraints $f_2(\alpha) \le 42.0$ and $f_3(\alpha) \le 32.0$; the rest of functional constraints $f_1(\alpha) \le 1100.0$, $33.0 \le f_2(\alpha)$ and $27.0 \le f_3(\alpha)$ are "rigid". According to the PSI method, in order to define "soft" constraints on functional relations f_2 and f_3, the latter should be interpreted as the *pseudo-criteria*, i.e. $\Phi_1 = f_2$ and $\Phi_2 = f_3$.

The system is to be minimized with respect to the following four performance criteria:

- $\Phi_3 = X_{1\partial}$ (mm)—vibration amplitude of the first mass;
- $\Phi_4 = M_1 + M_2$ (kg)—metal consumption of the system;
- $\Phi_5 = X_{1\partial}/X_{1\text{st}}$—dimensionless dynamical characteristic of the system, where $X_{1\text{st}}$ is the static displacement of mass M_1 under the action of the force P;
- $\Phi_6 = \omega/p_1$—dimensionless dynamical characteristic of the system.

Because criteria are antagonistic, there are difficulties in defining the criteria constraints correctly.

Taking into account the aforesaid, we formulate the initial optimization problem. We have a vector of criteria

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6), \tag{4}$$

on the basis of which test tables will be constructed and criteria constraints will be defined. Furthermore, we have design variable constrains (2) and three "rigid" functional constraints

$$1100 \ge f_1(\alpha), \qquad 33.0 \le f_2(\alpha) \quad \text{and} \quad 27.0 \le f_3(\alpha). \tag{3a}$$

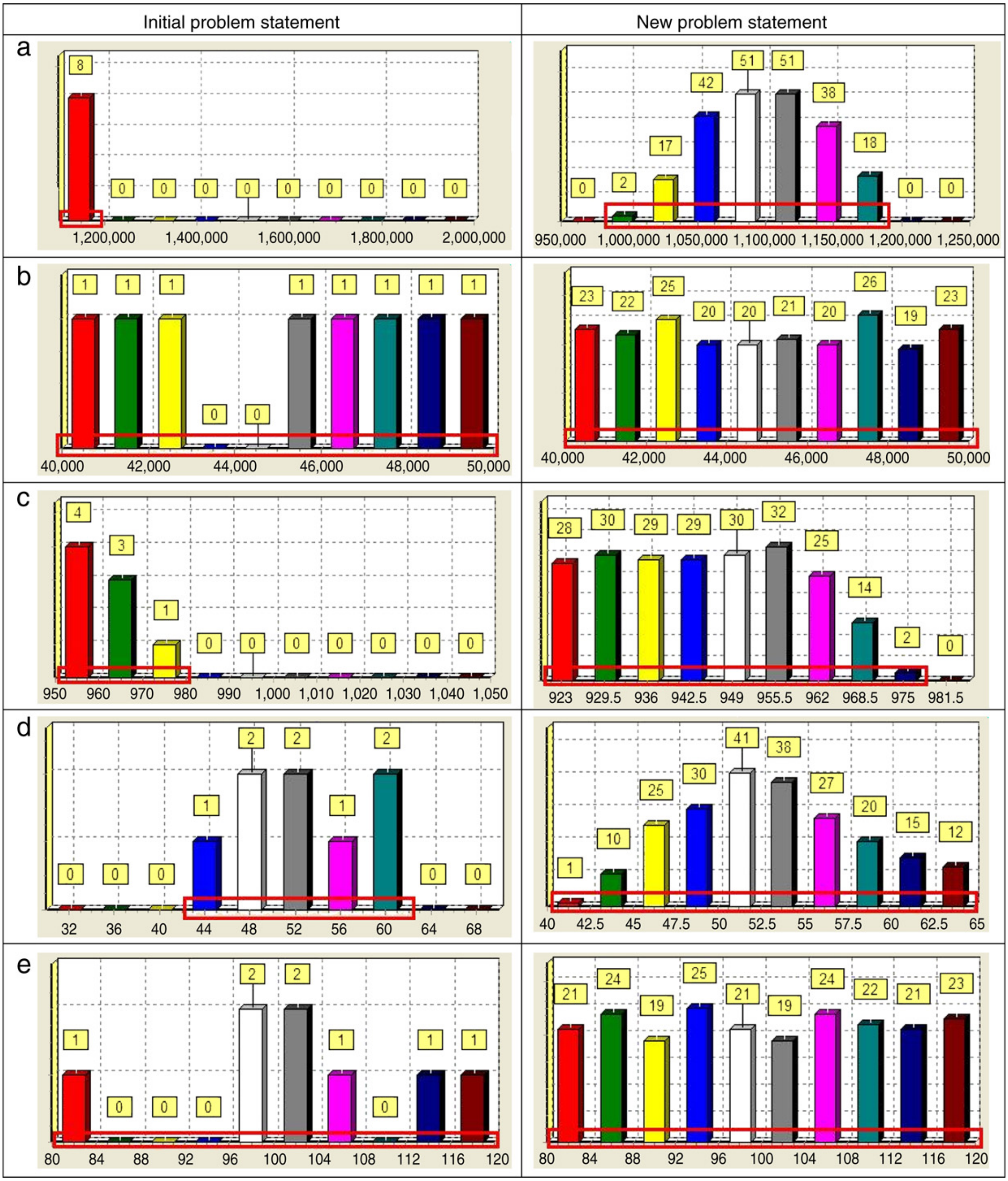

Fig. 2. Initial problem statement (left column) and new problem statement (right column): Histograms of distribution of the feasible and Pareto optimal solutions for the first (a), second (b), third (c), fourth (d), and fifth (e) design variables.

3.2. Definition of feasible and Pareto optimal sets in the initial problem statement

We carried out 1024 trials in a design variable space. Each trail point corresponds to the design variable vector. These vectors were generated with uniformly distributed sequences (LP_τ sequences). 765 vectors have satisfied given design variable and "rigid" functional constraints. After dialogs of expert with computer, the criteria constraints were determined. As a result, only 8 feasible solutions, including 6 Pareto optimal solutions, satisfied all constraints. The PSI method provides information about the distribution of feasible solutions in design variable space via histograms [8]. The intervals of each

Table 1
Definition of boundaries for the first, third and fourth design variables in the new problem.

Initial intervals of variation of design variables (Initial Problem)	Subintervals where the feasible solutions belong (Initial Problem)	New intervals of variation of design variables (New Problem)
$1.1 \times 10^6 \le \alpha_1 \le 2.0 \times 10^6$	$1.1 \times 10^6 \le \alpha_1 \le 1.17 \times 10^6$	$9.25 \times 10^5 \le \alpha_1 \le 1.25 \times 10^6$
$950 \le \alpha_3 \le 1050$	$951 \le \alpha_3 \le 975$	$920 \le \alpha_3 \le 985$
$30 \le \alpha_4 \le 70$	$42.7 \le \alpha_4 \le 60.35$	$40 \le \alpha_4 \le 65$

design variable are divided into 10 identical subintervals, see Fig. 2. Above each subinterval, the number of feasible solutions entering this subinterval is indicated. The region of the feasible solutions is marked with a red rectangle (Fig. 2).

The histograms show the range of change of each design variable and location of feasible solutions in the corresponding intervals. In particular, feasible solutions for the first and third design variables are located in the left ends of the interval; and for the fourth design variable they are located in the middle of the interval. On the other hand, the feasible solutions for the second and fifth design variables are more or less uniformly distributed along the interval (Fig. 2, left column). Analysis of the histograms allows correction of constraints, e.g. new constraints can be defined that correspond to intervals where most feasible solutions belong. Other visualization tools for multicriteria analysis are described in [9].

3.3. New problem statement

Analyzing obtained results allowed us to change constraints for the first, third, and fourth design variables to focus on the regions where most feasible solutions belong, see Fig. 2a, b, c and Table 1. The rest of the design variable, functional and criteria constraints remained unchanged. Again, 1024 trials were conducted. As result, the number of feasible and a Pareto optimal solutions has increased to 219 and 25, respectively. Histograms for the new problem are shown in Fig. 2 (right column).

3.4. Combined Pareto optimal set

Given results from the above two problem statements, the combined Pareto optimal set was defined on the combined feasible solution set. The combined Pareto optimal set contains 25 solutions that belong only to the new problem statement. Thus, all Pareto optimal solutions from the initial problem statement were improved. This emphasizes the sensitivity of the feasible and Pareto optimal sets to changes in the constraints.

4. Conclusion

Correct determination of design variable, functional, and criteria constraints is a major challenge in real-life optimization problems. The most promising solution approach involves two stages. In the first stage, the feasible solution and Pareto optimal sets are constructed and analyzed. These sets are constructed on the basis of the PSI method. Analysis of the feasible solution set shows the work of all constraints; the cost of making concessions in various constraints, i.e. what are the losses and the gains; expediency of modification of constraints; and resources for improvement of the object by all criteria. Only after the first stage one can make a decision as to whether it is necessary to improve the obtained results by means of various optimization methods, including stochastic, genetic, and so on [10].

Finally, we refer the reader to works [2–4,8,9] that discuss in detail management of "soft" constraints in real-life problems of design and identification of the automobile, airplane, ship, machine tools, nuclear reactor, multipurpose airspace systems, and multistage axial flow compressor for the aircraft engine.

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