Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Computers and Mathematics with Applications 52 (2006) 1–32
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Multicriteria Analysis Tools in Real-Life Problems

R. STATNIKOV
Department of Information Science, Naval Postgraduate School
Monterey, CA 93943, U.S.A.
and
Mechanical Engineering Research Institute
Russian Academy of Sciences, Moscow 101990, Russia
rstatnik@nps.edu

A. BORDETSKY
Department of Information Science, Naval Postgraduate School
Monterey, CA 93943, U.S.A.
abordets@nps.edu

A. STATNIKOV*
Department of Biomedical Informatics, Vanderbilt University
Nashville, TN 37232, U.S.A.
alexander.statnikov@vanderbilt.edu

I. YANUSHKEVICH
Mechanical Engineering Research Institute
Russian Academy of Sciences, Moscow 101990, Russia
ivyanushkevich@farm-std.ru

Abstract—Applied optimization problems such as design, identification, design of controlled systems, operational development of prototypes, analysis of large-scale systems, and forecasting from observational data are multicriteria problems in essence. Construction of the feasible solution set is of primary importance in the above problems. The definition of a feasible solution set is usually considered to be the skill of a designer. Even though this skill is essential, it is by no means sufficient for the correct statement of the problem. There are many antagonistic performance criteria and all kinds of constraints in these problems; therefore, it is quite difficult to correctly determine the feasible set. As a result, ill-posed problems are solved, and optimal solutions are searched for far from where they should be. As a consequence, the optimization results have no practical meaning. In this work we propose methods and tools that will assist the designer in defining the feasible solution set correctly.

Keywords—Parameter space investigation (PSI) method, Multicriteria analysis, Multicriteria problems, Uniformly distributed sequences, Feasible solution set.

1. INTRODUCTION

For the constructing of the feasible solution set, a method called the parameter space investigation (PSI method) has been created and successfully integrated into various fields of industry, science,

*Author to whom all correspondence should be addressed.

0898-1221/06/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2006.08.002

and technology [1-12]. The PSI method most fully meets important features of engineering optimization problems. This method has been used in designing the space shuttle, nuclear reactors, unmanned vehicles, cars, ships, metal-tools, etc. The PSI method is based on the systematic search in multidimensional parameter space by using uniformly distributed sequences. Another key feature of the method is an interactive dialogue of a designer with the computer providing valuable information about improvement of the basic criteria. In other words, the designer can analyze potential gains and losses in making concessions for various criteria. The PSI method is implemented in the MOVI (multicriteria optimization and vector identification) software that can be executed on a standard PC [13].

The purpose of the present paper is to demonstrate techniques of correct construction of the feasible solution set, multicriteria analysis tools provided by using MOVI, and their applications for statement and solution of applied optimization problems. Many of the above problems until recently appeared to be intractable.

The method for constructing and analyzing the feasible solution set presented in this paper is oriented towards the statement and solution of real-life optimization problems. However, for the purposes of demonstrating the potentialities of the PSI method, we will also consider some relatively simple mathematical models. Despite their simplicity, the search for optimal solutions by traditional methods presents great, sometimes insurmountable difficulties. We would like to draw attention to the importance and complexity of multicriteria analysis, especially as it concerns real-life problems.

This paper is organized as follows: formulation and solution of multicriteria optimization problems are discussed in Section 2. Some basic features of engineering optimization problems are described in that section as well. The need to use uniformly distributed sequences to investigate the design variable space is demonstrated in Section 3. The PSI method as a tool for formulating and solving engineering optimization problems is presented in Section 4. The principal tools of multicriteria analysis based on the MOVI software are presented in Section 5. Multicriteria design, one of the fundamental engineering optimization problems, is described in Section 6. Construction and analysis of the feasible sets for solving this problem are also discussed in this section. In many cases, it is necessary to carry out a large-scale numerical experiment in order to construct the feasible solution set. The MOVI program makes it possible to solve these problems in parallel mode (Section 7). This section also discusses carrying out large-scale numerical experiments in problems with approximate models. Finally, we consider solutions of other important applied problems using the PSI method and MOVI software (Section 8): multicriteria optimal design of controlled engineering systems, multicriteria identification, operational development of prototypes, and multicriteria analysis from observational data.

2. FORMULATION AND SOLUTION OF MULTICRITERIA OPTIMIZATION PROBLEMS

Let us consider an object whose operation is described by a system of equations (differential, algebraic, etc.) or whose performance criteria can be directly calculated. We assume that the system depends on r design variables $\alpha_1, \dots, \alpha_r$ representing a point $\alpha = (\alpha_1, \dots, \alpha_r)$ in the r-dimensional space. In the general case one has to take into account design variable, functional, and criteria constraints. The *design variable constraints* have the form

$$\alpha_j^* \le \alpha_j \le \alpha_j^{**}, \qquad j = 1, \dots, r. \tag{1}$$

The *functional constraints* can be written as follows:

$$C_l^* \le f_l(\alpha) \le C_l^{**}, \qquad l = 1, \dots, t, \tag{2}$$

where the *functional relationships* $f_l(\alpha)$ may be either implicit or explicit functions of α, and C_l^* and C_l^{**} are the lower and the upper admissible values of the quantity $f_l(\alpha)$, respectively. The operation of the object is described by the particular performance criteria $\Phi_v(\alpha)$, $v = 1, \dots, k$. All other things being equal, it is desired that these criteria $\Phi_v(\alpha)$ be optimized. For simplicity, we assume that functions $\Phi_v(\alpha)$ are to be minimized.

The constraints (1) single out a parallelepiped Π in the r-dimensional design variable space. In turn, constraints (1) and (2) together define a certain subset G in Π. To avoid situations in which the designer regards the values of some criteria as unacceptable, we introduce *criteria constraints* in the form

$$\Phi_v(\alpha) \leq \Phi_v^{**}, \qquad v = 1, \dots, k, \tag{3}$$

where Φ_v^{**} is the worst value of criterion $\Phi_v(\alpha)$ acceptable to the designer. (The choice of Φ_v^{**} is discussed in Section 4.)

Criteria constraints differ from the functional constraints in that the former are adjusted while solving a problem and as a rule are repeatedly revised. Hence, unlike C_l^* and C_l^{**}, reasonable values of Φ_v^{**} cannot be chosen before solving the problem.

Constraints (1)–(3) define a feasible solution set D, i.e., a set of design solutions that satisfy the constraints, and hence, $D \subset G \subset \Pi$. In a prior work [1], we have demonstrated that if functions $f_l(\alpha)$ and $\Phi_v(\alpha)$ are continuous in Π, then the sets G and D are closed.

Let us formulate one of the basic problems of multicriteria optimization. It is necessary to find a set $P \subset D$ such that

$$\Phi(P) = \min_{\alpha \in D} \Phi(\alpha), \tag{4}$$

where $\Phi(\alpha) = (\Phi_1(\alpha), \dots, \Phi_k(\alpha))$ is the criterion vector and P is the Pareto optimal set.

We mean that $\Phi(\alpha) < \Phi(\beta)$ if for all $v = 1, \dots, k$, $\Phi_v(\alpha) \leq \Phi_v(\beta)$ and for at least one $v_0 \in \{1, \dots, k\}$, $\Phi_{v_0}(\alpha) < \Phi_{v_0}(\beta)$.

DEFINITION. *A point $\alpha^0 \in D$ is called the Pareto optimal point if there exists no point $\alpha \in D$ such that $\Phi_v(\alpha) \leq \Phi_v(\alpha^0)$ for all $v = 1, \dots, k$ and $\Phi_{v_0}(\alpha) < \Phi_{v_0}(\alpha^0)$ for at least one $v_0 \in \{1, \dots, k\}$. A set $P \subset D$ is called a Pareto optimal set if it consists of Pareto optimal points.*

In solving the above problem, one still has to determine the vector of design variables $\alpha^0 \in P$, which is the most preferred among the vectors belonging to set P.

2.1. Some Basic Features of Engineering Optimization Problems

Many engineering optimization problems share the following features:

- The problems are essentially multicriteria ones. As a rule, attempts are made to reduce multicriteria problems to single-criterion problems. These numerous attempts to construct a generalized criterion in the form of a combination of particular criteria have proved to be fruitless.
- The determination of the feasible solution set is one of the fundamental issues of the analysis of engineering problems. The construction of this set is an important step in the formulation and solution of such problems.
- Problem formulation and solution comprise a single process. The customary approach is that the designer first states the problem and then a computer is employed to solve it. This approach is untenable, since one can rarely formulate a problem completely and correctly before solving it. Thus, problems should be formulated and solved interactively.
- As a rule, mathematical models are complex systems of equations (including differential and other types of equations) that may be linear or nonlinear, deterministic or stochastic, with distributed or lumped parameters. Sometimes mathematical models have to be derived from observational data using machine learning techniques.
- The feasible solution set can be multiply connected, and its volume may be several orders of magnitude smaller than that of the domain within which the optimal solution is sought.
- Both the feasible solution set and the Pareto optimal set are nonconvex. In the general case, as a rule, information about the smoothness of criteria is not present. These criteria

functions are usually nonlinear and continuous; however, they may be nondifferentiable as well.

- A typical problem may contain a large number of constraints, and the dimensionality of the design variable and the criterion vectors may reach many dozens.
- The analysis of the feasible set is of importance for designers. It allows one to not only correct the initial boundaries of the design variable ranges, but also to revise the original mathematical models and criteria.
- A large-scale numerical experiment is often required in order to solve many real-life problems.
- Designers do not very often encounter serious difficulties in analyzing the feasible solution set and Pareto optimal set and in choosing the most preferred solution. They have a sufficiently well-defined system of preferences. Moreover, the aforementioned sets usually contain a small number of elements.

3. UNIFORMLY DISTRIBUTED SEQUENCES IN MULTIDIMENSIONAL DOMAINS

The features of the problems under consideration make it necessary to represent vectors α by points of uniformly distributed sequences in the design variable space [1–5]. We briefly summarize this approach below.

For many applied problems, the following situation is typical. There exists a multidimensional domain in which a function or a system of functions is considered whose values are calculated at certain points. Suppose that we wish to obtain some information on the behavior of the function in the entire domain or in a subdomain. Then, in the absence of additional information about the function, it is natural to require the points where the function is calculated to be uniformly distributed in some sense within the domain. Suppose that we consider a sequence of points $P_1, P_2, \dots, P_i, \dots$ belonging to a unit r-dimensional cube K^r. We denote by G an arbitrary domain in K^r and we denote by $S_N(G)$ the number of points P_i belonging to G $(1 \le i \le N)$. The sequence P_i is called *uniformly distributed* in K^r, if

$$\lim_{N\to\infty} \frac{S_N(G)}{N} = V_G, \tag{5}$$

where $V(G)$ is the volume of the r-dimensional domain G. If, instead of the unit cube, a parallelepiped Π is considered, then the right-hand side of (5) transforms into $V(G)/V(\Pi)$.

The meaning of the definition is the following [1,3]: for large values of N, the number of points of a given sequence belonging to an arbitrary domain G is proportional to volume $V(G)$,

$$S_N(G) \sim NV(G). \tag{6}$$

In solving engineering problems, one must commonly deal not with K^r, but with a certain parallelepiped Π, and, hence, move from the coordinates of the points uniformly distributed in K^r to those in Π.

Let us formulate the following statements [1]: if points Q_i with Cartesian coordinates $(q_{i1}, \dots, q_{ir})$ form a uniformly distributed sequence in K^r, then points α^i with Cartesian coordinates $\alpha_1^i, \dots, \alpha_r^i$, where

$$\alpha_j^i = \alpha_j^* + q_{ij}\left(\alpha_j^{**} - \alpha_j^*\right), \qquad j = 1, 2, \dots, r, \tag{7}$$

form a uniformly distributed sequence in parallelepiped Π consisting of points $(\alpha_1, \dots, \alpha_r)$ whose coordinates satisfy the inequalities $\alpha_j^* \le \alpha_j \le \alpha_j^{**}$.

Let $\alpha^1, \dots, \alpha^i, \dots$ be a sequence of points uniformly distributed in Π, and $G \subset \Pi$ be an arbitrary domain with volume $V(G) > 0$. If among the points α^i, one chooses all the points belonging to G, then one obtains the sequence of points uniformly distributed in G [1].

3.1. Quantitative Characteristics of Uniformity

Let us fix a net consisting of the points $P_1, \dots, P_N \in K$. To estimate the uniformity of distribution of these points quantitatively, we introduce the quantity $D(P_1, \dots, P_N)$ called the *discrepancy*, implying the discrepancy between the 'ideal' and actual uniformities.

Let P be an arbitrary point belonging to K and G_P be an n-dimensional parallelepiped with the diagonal OP and faces parallel to the coordinate planes (Figure 1). Denote by V_{G_P} the volume of G_P and by $S_N(G_P)$, the number of points P_i which enter G_P and whose subscripts satisfy the inequalities $1 \le i \le N$.

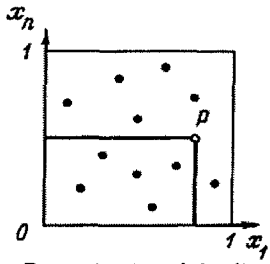


Figure 1. Determination of the discrepancy.

The discrepancy of the points $P_1, \dots, P_N$ is

$$D(P_1, \dots, P_N) = \sup_{P \in K} \left| S_N(G_P) - N V_{G_P} \right|, \tag{8}$$

where the supremum is taken over all possible positions of the point P in the cube. It is natural to consider that the smaller $D(P_1, \dots, P_N)$ is, the more uniformly the points $P_1, \dots, P_N$ are arranged. Among uniformly distributed sequences known at present, the so-called LP_τ sequences and P_τ nets are among the best ones in terms of uniformity characteristics as $N \to \infty$, see [1–5].

4. THE PARAMETER SPACE INVESTIGATION METHOD IS A TOOL FOR FORMULATING AND SOLVING ENGINEERING OPTIMIZATION PROBLEMS

In Section 2, we formulated the problem of multicriteria optimization and defined the feasible solution set D, which is constructed using the values of Φ_v^{**}, $v = 1, \dots, k$ and some other constraints. Now we proceed by describing the parameter space investigation (PSI) method, which allows correct determination of Φ_v^{**} and, hence, of the feasible solutions as well.

The PSI method consists of the following three stages.

STAGE 1. COMPILATION OF TEST TABLES WITH THE HELP OF A COMPUTER. First, using uniformly distributed sequences,[1] one chooses N trial points $\alpha^1, \dots, \alpha^N$, satisfying relation (2). Suppose that the designer can *a priori* indicate the constraints $\tilde{\Phi}_v^{**}$ to be imposed on the criteria $\Phi_v(\alpha)$, $v = 1, \dots, k'$. $\tilde{\Phi}_v^{**}$ is the value of the v^{th} criterion for which the values $\Phi_v(\alpha) > \tilde{\Phi}_v^{**}$ are known to be unacceptable. The constraints $\tilde{\Phi}_v^{**}$, if any, should be imposed successively. First, one should calculate $\Phi_1(\alpha^i)$. If the inequality $\Phi_1(\alpha^i) \le \tilde{\Phi}_1^{**}$ is satisfied, then we proceed to the calculation of the criterion $\Phi_2(\alpha^i)$, and so on. The vectors α^i violating this inequality are discarded. Finally, only the vectors α^i satisfying all constraints C_i^*, C_i^{**}, and $\tilde{\Phi}_v^{**}$ will remain. Then, for each of the k' criteria a test table[2] is compiled so that the values of $\Phi_v(\alpha^1), \dots, \Phi_v(\alpha^N)$ are arranged in increasing order, i.e.,

$$\Phi_v\left(\alpha^{i_1}\right) \le \Phi_v\left(\alpha^{i_2}\right) \le \dots \le \Phi_v\left(\alpha^{i_N}\right), \qquad v = 1, \dots, k', \quad k' \le k, \tag{9}$$

where $i_1, i_2, \dots, i_N$ are the numbers of trials (a separate set for each v).

[1] Very often LP_τ sequences are applied for these purposes. See also Section 5.2.

[2] Sometimes it is called an *ordered test table*, for example, see Figure 2. In an unordered table, the columns are formed of the values of $\Phi_v(\alpha^i)$, $i = 1, \dots, N$, $v = 1, \dots, k$.

The remaining criteria $\Phi_v(\alpha)$, $v = k'+1,\dots,k$ should be calculated only for the vectors satisfying all inequalities of (9). By analogy with the criteria $\Phi_v(\alpha)$, $v = 1,\dots,k'$ test tables are constructed for the criteria $\Phi_v(\alpha)$, $v = k'+1,\dots,k$. Taken together, the k tables form a complete test table.

STAGE 2. SELECTION OF CRITERIA CONSTRAINTS. This stage requires intervention of the designer. When successively analyzing inequalities (9), the designer specifies the criteria constraints Φ_v^{**}. Note that the method described is in practice convenient for a designer. Actually, the designer has to consider one criterion at a time and specify the respective constraint. The designer should not "balance" by reducing some criterion at the expense of the others: one analyzes one test table and imposes the criterion constraint. Then one proceeds to the next table, and so on. Note that the revision of the criteria constraints within the limit of the test tables that have been constructed does not lead to any difficulties for the designer.

All Φ_v^{**} should be the maximum values of the criteria $\Phi_v(\alpha)$, which guarantee an acceptable level of the object's operation. If the selected values of Φ_v^{**} are not a maximum, then many important solutions may be lost, since some of the criteria may be contradictory. Note that when solving practical problems, the designer often cannot determine the maximum values of criteria constraints.

As a rule, the designer may set Φ_v^{**} equal to a criterion value $\Phi_v(\bar{\alpha})$ whose feasibility is beyond doubt.

STAGE 3. VERIFICATION OF THE SOLVABILITY OF PROBLEM (4) WITH THE HELP OF A COMPUTER. Let us fix a criterion, say $\Phi_{v_1}(\alpha)$, and consider the corresponding table (see (9)), and let S_1 be the number of values in the table satisfying the selected criterion constraint

$$\Phi_{v_1}\left(\alpha^{i_1}\right) \le \cdots \le \Phi_{v_1}\left(\alpha^{i_{S_1}}\right) \le \Phi_{v_1}^{**} = \Phi_{v_1}\left(\bar{\alpha}\right). \tag{10}$$

One should choose the value of criterion Φ_{v_1} for which S_1 is minimum among the analogous numbers calculated for each of the criteria Φ_v.

Then the value of criterion Φ_{v_2} is selected by analogy with Φ_{v_1}, and the values of $\Phi_{v_2}(\alpha^{i_1}),\dots,$ $\Phi_{v_2}(\alpha^{i_{S_1}})$ in the test table are considered. Let the table contain $S_2 \le S_1$ values such that $\Phi_{v_2}(\alpha^{i_j}) \le \Phi_{v_2}^{**}$, $1 \le j \le S_2$. Similar procedures are carried out for each of the criteria. Then if at least one point can be found for which all inequalities (3) are valid simultaneously, the set D defined by inequalities (1)–(3) is nonempty and problem (4) is solvable; i.e., $D \ne \emptyset$. Otherwise $D = \emptyset$, and one should return to Stage 2 and ask the designer to make certain concessions in the specification of Φ_v^{**}. However, if the concessions are highly undesirable, then one may return to Stage 1 and increase the number of points in order to repeat Stages 2 and 3 using the extended test table.

The procedure is to be continued until D is nonempty and the designer finds the acceptable solutions. Otherwise, the designer can attempt to improve these solutions by returning to Stage 1 and/or Stage 2. The Pareto optimal set P is then constructed in accordance with the definition presented in Section 2. This is done by removing those feasible points that can be improved with respect to all the criteria simultaneously.

Let us describe the procedure for constructing the maximum feasible set. If the selected values of $\Phi_v^{**} = \Phi_v(\bar{\alpha})$ are not the maximum ones, then one is not sure whether the values of $\Phi_v(\alpha)$ from the interval $\Phi_v(\bar{\alpha}) \le \Phi_v(\alpha) \le \tilde{\Phi}_v^{**}$ are feasible or not. In this case one has to construct the feasible solution set D under the constraints $\Phi_v^{**} = \Phi_v(\bar{\alpha})$ and the corresponding Pareto optimal set P. Further, the set $\tilde{D}$ is constructed under the constraints $\tilde{\Phi}_v^{**}$, $v = 1,\dots,k$, as well as the corresponding Pareto optimal set $\tilde{P}$. Let us compare $\Phi(P)$ and $\Phi(\tilde{P})$. If the vectors belonging to $\Phi(\tilde{P})$ do not substantially improve the value of the vectors from $\Phi(P)$, then one may set $\Phi_v^{**} = \Phi_v(\bar{\alpha})$. Otherwise, if the improvement is significant, then the values of the criteria constraints may be set equal to $\tilde{\Phi}_v^{**}$. In this case one has to make sure that the optimal solutions thus obtained are feasible. If the designer is unable to do this, then the criteria constraints are set equal to their previous values,

$$\Phi_v^{**} = \Phi_v(\bar{\alpha}).$$

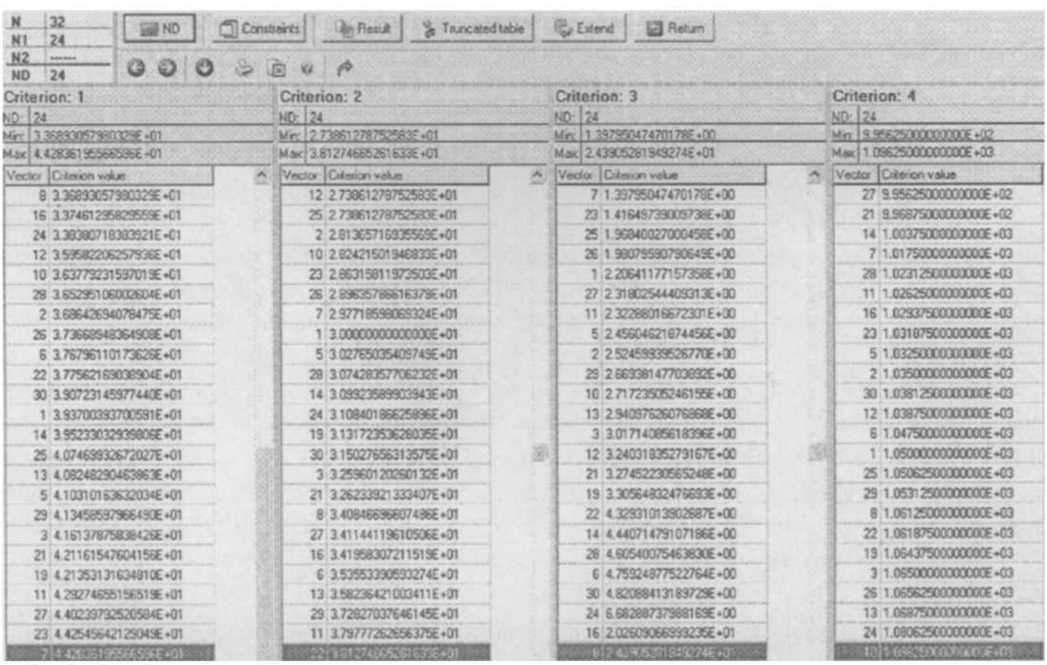
This scheme can be used for all possible values of $\Phi_v(\bar{\alpha})$ and $\tilde{\Phi}_v^{**}$.

The PSI method has proved to be a very convenient and effective tool for the designer, primarily because this method can be directly used for the statement and solution of the problem in an interactive mode.

The problems of approximating the feasible solution set and Pareto optimal set are considered in [5].

4.1. Example of Test Tables

Test tables with four criteria after 32 trials ($N = 32$) are presented in Figure 2. The test tables are obtained using MOVI software (see Section 5). Recall that for each criterion there is a corresponding test table (column). The table contains 24 vectors ($N1 = 24$); the remaining eight were not included in the test table, since they did not satisfy the functional constraints. All solutions are arranged in the tables in the order of deteriorating values of the performance criteria. For example, for the first criterion, vector 8, with a criterion value of 33.6893, is the best. Next in the order of deteriorating value of the first criterion are vectors $16, 24, 12, \ldots$. The worst is vector 7, with a value of this criterion equal to 44.2836. The minimum (33.6893) and maximum (44.2836) criterion values among the solutions entering the test table are shown above the table. The remaining three tables are constructed in a similar manner. For the second criterion, vectors 12 and 25 are the best and vector 22 is the worst; in the third criterion, vector 7 is the best and vector 8 is the worst. It can be seen from the table that the first and third criteria are contradictory: the best solutions in the first criterion are the worst in the third criterion and vice versa. Criteria constraints Φ_1^{**}, Φ_2^{**}, Φ_3^{**}, Φ_4^{**} are highlighted by dark lines; they correspond to vectors 7, 22, 8, and 10. Since the worst values of the criteria are taken as the constraints, the number of feasible solutions $ND = 24$.

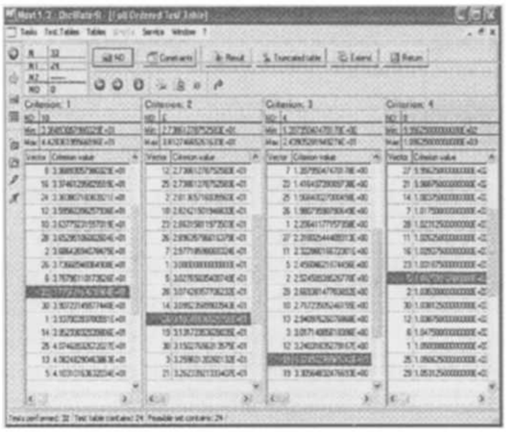


Criterion: 1		Criterion: 2		Criterion: 3		Criterion: 4	
ND:	24	ND:	24	ND:	24	ND:	24
Min:	3.36893057980329E+01	Min:	2.73861278752583E+01	Min:	1.39795047470178E+00	Min:	9.95625000000000E+02
Max:	4.42836195566596E+01	Max:	3.81274685261633E+01	Max:	2.43905281949274E+01	Max:	1.09625000000000E+03
Vector	Criterion value	Vector	Criterion value	Vector	Criterion value	Vector	Criterion value
8	3.36893057980329E+01	12	2.73861278752583E+01	7	1.39795047470178E+00	27	9.95625000000000E+02
16	3.37461295829559E+01	25	2.73861278752583E+01	23	1.41649739009738E+00	21	9.96875000000000E+02
24	3.38380718383921E+01	2	2.81365716935569E+01	25	1.96840027000458E+00	14	1.00375000000000E+03
12	3.59582206257936E+01	10	2.82421501946833E+01	26	1.98079590790649E+00	7	1.01750000000000E+03
10	3.63779231597019E+01	23	2.86315811973503E+01	1	2.20641177157358E+00	28	1.02312500000000E+03
28	3.65295106002604E+01	26	2.89635786616379E+01	27	2.31802544409313E+00	11	1.02625000000000E+03
2	3.68642694078475E+01	7	2.97718598069324E+01	11	2.32288016672301E+00	16	1.02937500000000E+03
26	3.73668948364908E+01	1	3.00000000000000E+01	5	2.45604621874456E+00	23	1.03187500000000E+03
6	3.76796110173626E+01	5	3.02765035409749E+01	2	2.52459939526770E+00	5	1.03250000000000E+03
22	3.77562169038904E+01	28	3.07428357706232E+01	29	2.66938147703892E+00	2	1.03500000000000E+03
30	3.90723145977440E+01	14	3.09923589903943E+01	10	2.71723505246155E+00	30	1.03812500000000E+03
1	3.93700393700591E+01	24	3.10840186625896E+01	13	2.94097626076868E+00	12	1.03875000000000E+03
14	3.95233032939806E+01	19	3.13172353628035E+01	3	3.01714085618396E+00	6	1.04750000000000E+03
25	4.07469932672027E+01	30	3.15027656313575E+01	12	3.24031835279167E+00	1	1.05000000000000E+03
13	4.08248290463863E+01	3	3.25960120260132E+01	21	3.27452230565248E+00	25	1.05062500000000E+03
5	4.10310163632034E+01	21	3.26233921333407E+01	19	3.30564832476693E+00	29	1.05312500000000E+03
29	4.13458597966490E+01	8	3.40846696607486E+01	22	4.32931013902687E+00	8	1.06125000000000E+03
3	4.16137875838426E+01	27	3.41144119610506E+01	14	4.44071479107186E+00	22	1.06187500000000E+03
21	4.21161547604156E+01	16	3.41958307211519E+01	28	4.60540075463830E+00	19	1.06437500000000E+03
19	4.21353131634810E+01	6	3.53553390593274E+01	6	4.75924877522764E+00	3	1.06500000000000E+03
11	4.29274655156519E+01	13	3.58236421003411E+01	30	4.82088413189729E+00	26	1.06562500000000E+03
27	4.40239732520584E+01	29	3.72827037646145E+01	24	6.68288737988169E+00	13	1.06875000000000E+03
23	4.42545642129049E+01	11	3.79777262656375E+01	16	2.02609066999235E+01	24	1.08062500000000E+03
7	[illegible]	22	[illegible]	8	[illegible]	10	[illegible]

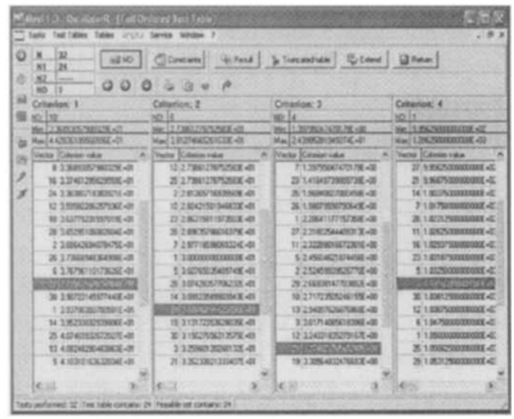
Figure 2. Example of test tables.

4.2. Dialogue of the Designer with the Computer

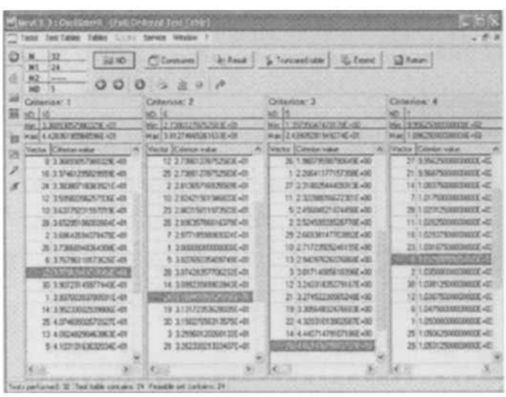
As it was already been mentioned, the dialogue of the designer with the computer is central to constructing the feasible set after carrying out N trials. We present four dialogues as examples. Criteria constraints Φ_1^{**}, Φ_2^{**}, Φ_3^{**}, Φ_4^{**} are shown as dark lines in Figure 3.



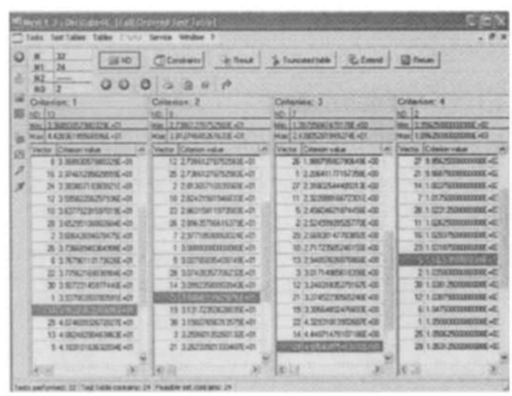
(a) Dialogue 1, $ND = 0$.



(b) Dialogue 2, $ND = 1$.



(c) Dialogue 3, $ND = 1$.



(d) Dialogue 4, $ND = 2$.

Figure 3. Examples of dialogues with the computer.

DIALOGUE 1. See Figure 3a. Criterion constraint $\Phi_1^{**} = 37.756$ is imposed on the first criterion. As a result, out of the 24 vectors entering the test table, 10 solutions satisfy this constraint (first column). After imposing the criterion constraint $\Phi_2^{**} = 31.084$, there are six feasible solutions (second column). After the criterion constraint $\Phi_3^{**} = 3.2745$, there are four feasible solutions (third column). After the criterion constraint $\Phi_4^{**} = 1032.5$, the feasible set is empty, $ND = \emptyset$ (fourth column). As it can be seen from Figure 3a, the criteria constraints in the first dialogue correspond to vectors 22, 24, 21, and 5. Since the feasible set is empty, one should either increase the number of trials N and/or revise the criteria constraints. Examples of dialogues where the criteria constraints were revised are given below.

DIALOGUE 2. See Figure 3b. The first three criteria constraints are the same as before, while a minor concession has been made in the fourth criterion: instead of $\Phi_4^{**} = 1032.5$, we have taken $\Phi_4^{**} = 1035$, which corresponds to vector 2. As a result, the feasible set contains one solution, vector 2, so $ND = 1$.

DIALOGUE 3. See Figure 3c. The first, second, and fourth criteria constraints are the same as in Dialogue 1. A concession has been made in the third criterion from $\Phi_3^{**} = 3.2745$ to $\Phi_3^{**} = 4.6054$, which corresponds to vector 28. The feasible set remains unchanged, and $ND = 1$.

DIALOGUE 4. See Figure 3d. Compared to Dialogue 1, concessions have been made in the first and third criteria to the values $\Phi_1^{**} = 39.523$ and $\Phi_3^{**} = 4.6054$. Two vectors have entered the feasible set, 14 and 28, and we have $ND = 2$.

5. IMPLEMENTATION OF THE PSI METHOD IN MOVI (MULTICRITERIA OPTIMIZATION AND VECTOR IDENTIFICATION)

MOVI, a comprehensive software system for multicriteria analysis, does not impose any limitations on the number of design variables and criteria; this number is bounded only by the technical characteristics of the computer. For many engineering optimization problems, the difficulty in determining the feasible set requires one to carry out a large-scale numerical experiment. MOVI allows these problems to be solved in parallel mode as described in Section 7. The flexible software architecture of MOVI allows optimization of mathematical models developed in Mathworks Matlab/Simulink, C/C++, and Borland Delphi.

5.1. Analysis Tools

The analysis tools provided in MOVI allow one to determine the functionality of the mathematical model and constraints, as well as provide hints for correcting the initial statement of the problem. Analysis tools include the following.

Tables of feasible and Pareto optimal solutions

After conducting the trails, MOVI provides the designer with information on the obtained results, that is, the values of the feasible and Pareto optimal criteria and design variable vectors. Analysis of these tables allows one to choose the most preferable solution to formulate new bounds for the design variables and investigate the new parallelepiped with the aim of improving previously found optimal solutions.

Histograms of feasible solutions

Visualization of the distribution of feasible solutions over the design variable intervals $[\alpha_j^*, \alpha_j^{**}]$, $j = 1, \dots, r$ is of great importance. In particular, the histograms show the role of the functional and criteria constraints in the design variable space and allow the designer to correct the initial design variable constraints accordingly.

Graphs criterion vs. design variable II

After N trials, $N1$ design variable vectors have entered the test table. We consider projections of the points $\Phi_v(\alpha^i)$, $v = 1, \dots, k$, $i = 1, \dots, N1$, onto the plane $\Phi_v \alpha_j$, $j = 1, \dots, r$. These graphs provide information on dependencies between criteria and design variables.

Graphs criterion vs. criterion

After N trials, $N1$ design variable vectors have entered the test table. We consider projections of the points $\Phi_v(\alpha^i)$, $v = 1, \dots, i, \dots, j, \dots, k$, $i = 1, \dots, N1$, onto the plane $\Phi_i \Phi_j$. These projections provide the designer with information about dependencies between criteria.

Graphs criterion vs. design variable I

After the analysis of the test table, preference was given to Pareto optimal vector α^i. We fix all components of this vector except for one, α_j^i, and find out how the changes of criteria $\Phi_1, \dots, \Phi_k$ depend on α_j^i in the initial interval $[\alpha_j^*; \alpha_j^{**}]$. This analysis is used for investigating Pareto optimal solutions.

Pseudocriteria and tables of the functional failures

The functional constraints are very often not specified rigidly; i.e., they may be revised in the process of solving the problem. However, it is very difficult to determine them correctly. As a result, we often obtain a "sparse" or even empty feasible solution set. Two means of determining soft functional criteria are considered in the PSI method. The first is to represent functional dependences $f_l(\alpha)$ in test tables in the form of pseudocriteria. Analysis of the test tables allows one to determine the constraints on the pseudocriteria with consideration of all criteria constraints. The second is correction of the initial values C_l^*, C_l^{**} using so-called tables of functional failures. Only those solutions that do not satisfy the functional constraints enter these tables. The purpose of the analysis of the tables of functional failures is to determine how the functional constraints "work" and to correct them if necessary.

The analysis of all tables, histograms, and graphs is an important process of correcting the initial statement of the problem, since when assigning *a priori* constraints, especially when there are many of them, the designer seldom knows how they will behave.

In Sections 6–8, we will show multicriteria analysis tools in action for solving the main classes of engineering optimization problems.

5.2 Various Generators for Systematic Search in the Design Variable Space

In [14], Steuer and Sun have called attention to the possibility of using random number generators (rng) in the PSI method (along with LP_τ sequences). We also mention here works by Halton [15], Hammersley [16], Hlawka [17], Faure [18], and Kuipers and Niederreiter [19,20], in which good uniformly distributed sequences (in the sense of the uniformity estimates) have been constructed. Furthermore, Statnikov and Matusov have noted that various pseudorandom sequences (nets) may be used in the PSI method [2–4,21].

Prior to solving a concrete problem, one cannot say with certainty which of the generators is most suitable. Much depends on the behavior of the criteria, the form of the functional and design variable constraints, the number of test trials, and the geometry of the feasible solution set. The foregoing primarily applies to problems where, for objective reasons, a small (insufficient) number of trials is conducted. Carrying out a small number of trials is characteristic of the investigation of real-life problems, for example, in problems with a high dimensionality of the design variable vector. We also consider problems that require a great deal of computer time to calculate one criteria vector. The investigation of finite element models can also be assigned to these operations. For these problems, various test trial generators—random number generator (rng) and other pseudorandom sequences—can be used in the PSI method.

The use of rng has turned out to be suitable for investigating multicriteria problems depending on many tens, hundreds, or thousands of design variables [21].

In addition to LP_τ sequences and rng, the MOVI allows the use of other generators. The possibility of using various generators in the PSI method for probing the design variable space makes the method even more versatile.

6. MULTICRITERIA DESIGN: CONSTRUCTION AND ANALYSIS OF THE FEASIBLE SETS

6.1. Two-Mass Dynamical System

In this example, we determine the feasible solution set of the two-mass dynamical system shown in Figure 4. The system consists of two bodies with masses M_1 and M_2. The mass M_1 is attached to a fixed base by a spring with stiffness coefficient K_1. A spring-and-dashpot element with stiffness coefficient K_2 and damping coefficient C is located between masses M_1 and M_2. The harmonic force $P \cdot \cos(\omega t)$ acts upon mass M_1. The amplitude and frequency of the exciting

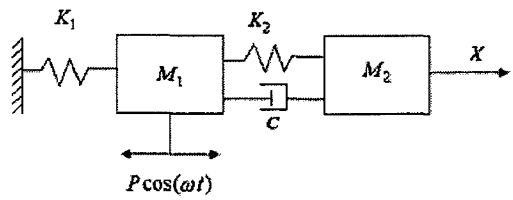


Figure 4. A two-mass dynamical system.

force are identified as $P = 2000(N)$ and $\omega = 30(s^{-1})$. The motion of this system is governed by the equations

$$\begin{aligned} M_1 X_1'' + C\,(X_1' - X_2') + K_1 X_1 + K_2\,(X_1 - X_2) &= P \cdot \cos(\omega t), \\ M_2 X_2'' + C\,(X_2' - X_1') + K_2 (X_2 - X_1) &= 0. \end{aligned} \tag{11}$$

We treat the parameters K_1, K_2, M_1, M_2, and C as the design variables to be determined, i.e., $\alpha_1 = K_1$, $\alpha_2 = K_2$, $\alpha_3 = M_1$, $\alpha_4 = M_2$, $\alpha_5 = C$. The design variable constraints are prescribed as the parallelepiped Π defined by the inequalities

$$\begin{aligned} 1.1 \cdot 10^6 \le \alpha_1 &\le 2.0 \cdot 10^6 \left(\frac{\mathrm{N}}{\mathrm{m}}\right); \\ 4.0 \cdot 10^4 \le \alpha_2 &\le 5.0 \cdot 10^4 \left(\frac{\mathrm{N}}{\mathrm{m}}\right); \\ 950 \le \alpha_3 &\le 1050\,(\mathrm{kg}); \\ 30 \le \alpha_4 &\le 70\,(\mathrm{kg}); \\ 80 \le \alpha_5 &\le 120 \left(\frac{\mathrm{N} \cdot \mathrm{s}}{\mathrm{m}}\right). \end{aligned} \tag{12}$$

There are three functional constraints (on the total mass and on the partial frequencies)

$$\begin{aligned} f_1(\alpha) &= \alpha_3 + \alpha_4 \le 1100.0\,(\mathrm{kg}); \\ 33.0 \le f_2(\alpha) &= p_1 = \sqrt{\frac{\alpha_1}{\alpha_3}} \le 42.0\,\left(\mathrm{s}^{-1}\right); \\ 27.0 \le f_3(\alpha) &= p_2 = \sqrt{\frac{\alpha_2}{\alpha_4}} \le 32.0\,\left(\mathrm{s}^{-1}\right). \end{aligned} \tag{13}$$

The upper limits imposed on the functions $f_2(\alpha)$ and $f_3(\alpha)$ are not rigid. For this reason, the functional relations $f_2(\alpha)$ and $f_3(\alpha)$ are interpreted as pseudocriteria Φ_1 and Φ_2. Thus, we have three functional constraints

$$\begin{aligned} f_1(\alpha) &= \alpha_3 + \alpha_4 \le 1100.0, \\ 33 &\le f_2(\alpha), \\ 27 &\le f_3(\alpha). \end{aligned} \tag{14}$$

We want to optimize the system with respect to the following four performance criteria

$\Phi_3 = X_{1\partial}$ (mm)—vibration amplitude of the first mass;

$\Phi_4 = M_1 + M_2$ (kg)—metal consumption of the system;

$\Phi_5 = \dfrac{X_{1\partial}}{X_{1^{\mathrm{st}}}}$ and $\Phi_6 = \dfrac{\omega}{p_1}$—dimensionless dynamical characteristics of the system,

where $X_{1^{\mathrm{st}}}$ is the static displacement of mass M_1 under the action of the force P. Thus, we have a vector of criteria $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6)$, which will be used for construction of the test tables. All criteria and pseudocriteria need to be minimized.

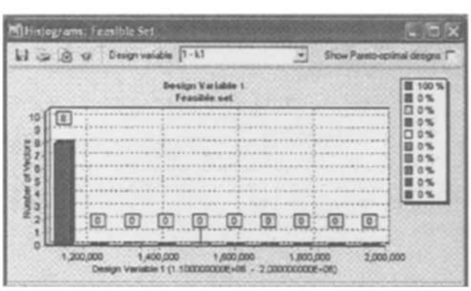
6.2. Experiments

EXPERIMENT 1. IS THE STATEMENT OF THE PROBLEM CORRECT? We performed 1024 trials[3] using LP_τ sequences and constructed the test table. A total of 789 solutions was included in the test table, since they satisfied the functional constraints. The remaining 235 solutions did not satisfy the functional constraints (14) and entered the *tables of functional failures*. While analyzing the test table, the following criteria constraints were formulated:

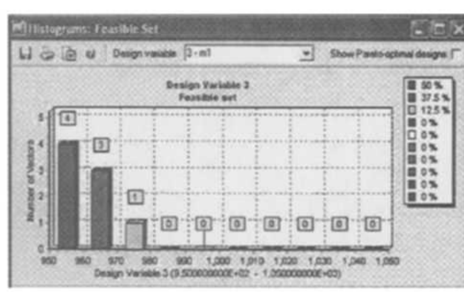
$$
\begin{aligned}
\Phi_1^{**} &= 35.2008;\\
\Phi_2^{**} &= 36.9807;\\
\Phi_3^{**} &= 8.4166;\\
\Phi_4^{**} &= 1019.1211;\\
\Phi_5^{**} &= 18.795;\\
\Phi_6^{**} &= 0.9087.
\end{aligned} \tag{15}
$$

Only eight solutions were found to be feasible (i.e., satisfied constraints (15)). Four of these feasible solutions are Pareto optimal (corresponding to trials #520, #336, #672, #288).

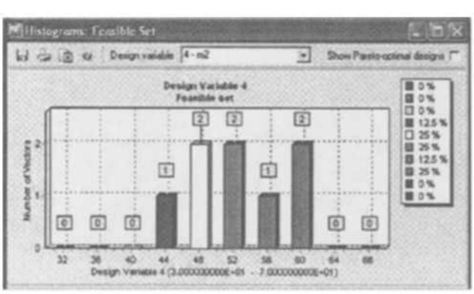
The analysis of the histograms shows the effect of design variable, functional and criteria constraints (see Figure 5). In particular, all feasible solutions for design variables α_1 and α_3 are located in the left ends of the intervals. The feasible solutions for the design variable α_4 are located in the middle of the interval. On the other hand, the feasible solutions for α_2 and α_5 are more or less uniformly distributed along the interval. These histograms were produced in the MOVI software system using the option *histograms of feasible solutions*. The results of analyzing the histograms for design variables α_1, α_3, and α_4 are summarized in Table 1. The first column of Table 1 lists the initial intervals of variation of α_1, α_3, and α_4. The second column contains



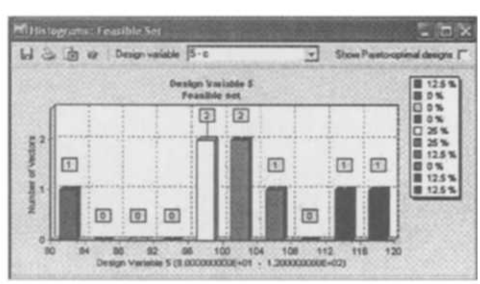
(a) First design variable.



(b) Third design variable.



(c) Fourth design variable.



(d) Fifth design variable.

Figure 5. Histograms of the distribution of feasible solutions.

[3]In this paper, we use equivalent words for describing *designs*: *solutions*, or *vectors*, or *trials*.

Table 1. Refining initial design variables constraints.

Initial Intervals of Variation of Design Variables (*Experiment 1*)	Subintervals Where the Feasible Solutions Belong (*Experiment 1*)	New Intervals of Variation of Design Variables (*Experiment 2*)
$1.1 \cdot 10^6 \leq \alpha_1 \leq 2.0 \cdot 10^6$	$1.1 \cdot 10^6 \leq \alpha_1 \leq 1.17 \cdot 10^6$	$9 \cdot 10^5 \leq \alpha_1 \leq 1.2 \cdot 10^6$
$950 \leq \alpha_3 \leq 1050$	$950 \leq \alpha_3 \leq 975$	$850 \leq \alpha_3 \leq 980$
$30 \leq \alpha_4 \leq 70$	$42 \leq \alpha_4 \leq 60.35$	$40 \leq \alpha_4 \leq 64$

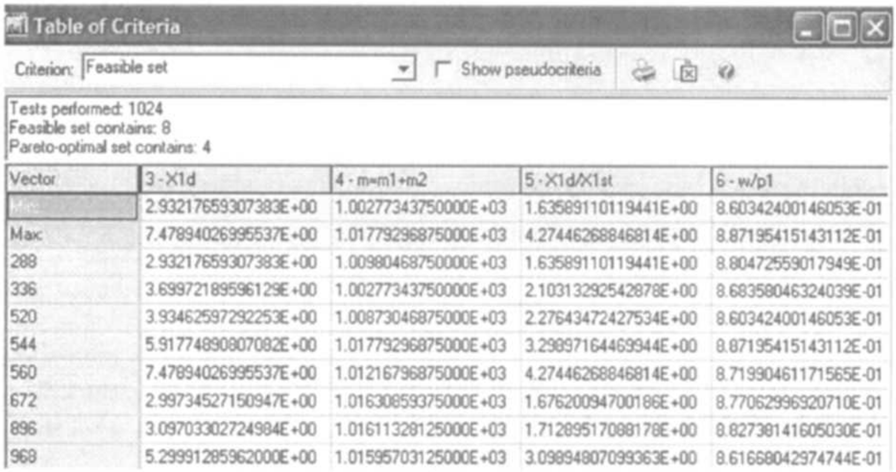


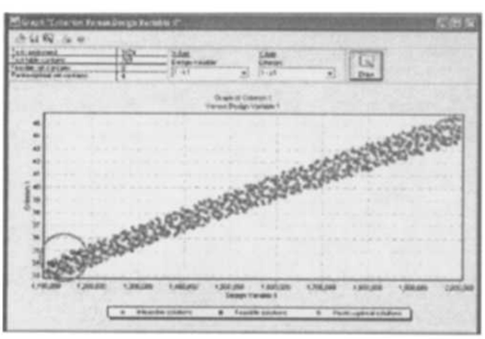
Table of Criteria

Criterion: Feasible set   Show pseudocriteria

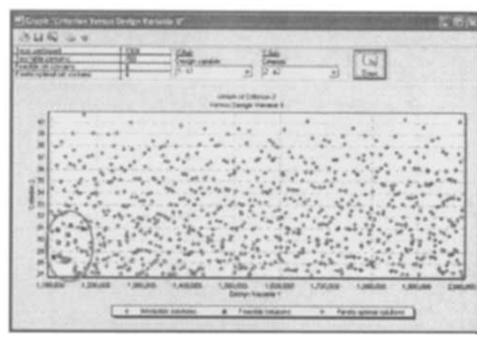
Tests performed: 1024
Feasible set contains: 8
Pareto-optimal set contains: 4

Vector	3 - X1d	4 - m=m1+m2	5 - X1d/X1st	6 - w/p1
[illegible]	2.93217659307383E+00	1.00277343750000E+03	1.63589110119441E+00	8.60342400146053E-01
Max	7.47894026995537E+00	1.01779296875000E+03	4.27446268846814E+00	8.87195415143112E-01
288	2.93217659307383E+00	1.00980468750000E+03	1.63589110119441E+00	8.80472559017949E-01
336	3.69972189596129E+00	1.00277343750000E+03	2.10313292542878E+00	8.68358046324039E-01
520	3.93462597292253E+00	1.00873046875000E+03	2.27643472427534E+00	8.60342400146053E-01
544	5.91774890807082E+00	1.01779296875000E+03	3.29897164469944E+00	8.87195415143112E-01
560	7.47894026995537E+00	1.01216796875000E+03	4.27446268846814E+00	8.71990461171565E-01
672	2.99734527150947E+00	1.01630859375000E+03	1.67620094700186E+00	8.77062996920710E-01
896	3.09703302724984E+00	1.01611328125000E+03	1.71289517088178E+00	8.82738141605030E-01
968	5.29991285962000E+00	1.01595703125000E+03	3.09894807099363E+00	8.61668042974744E-01

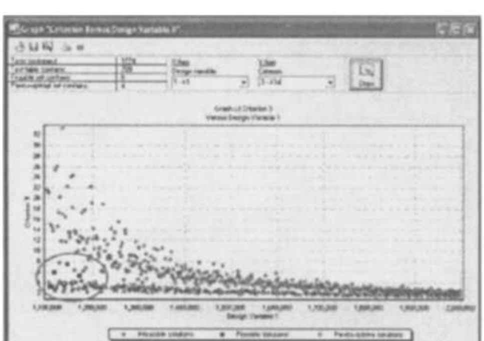
Figure 6. Feasible solutions (criteria vectors).



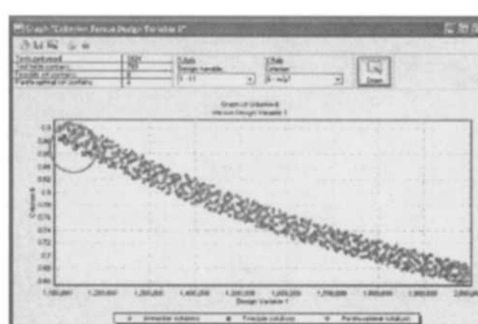
(a) Criterion 1 vs. design variable 1.



(b) Criterion 2 vs. design variable 1.



(c) Criterion 3 vs. design variable 1.



(d) Criterion 6 vs. design variable 1.

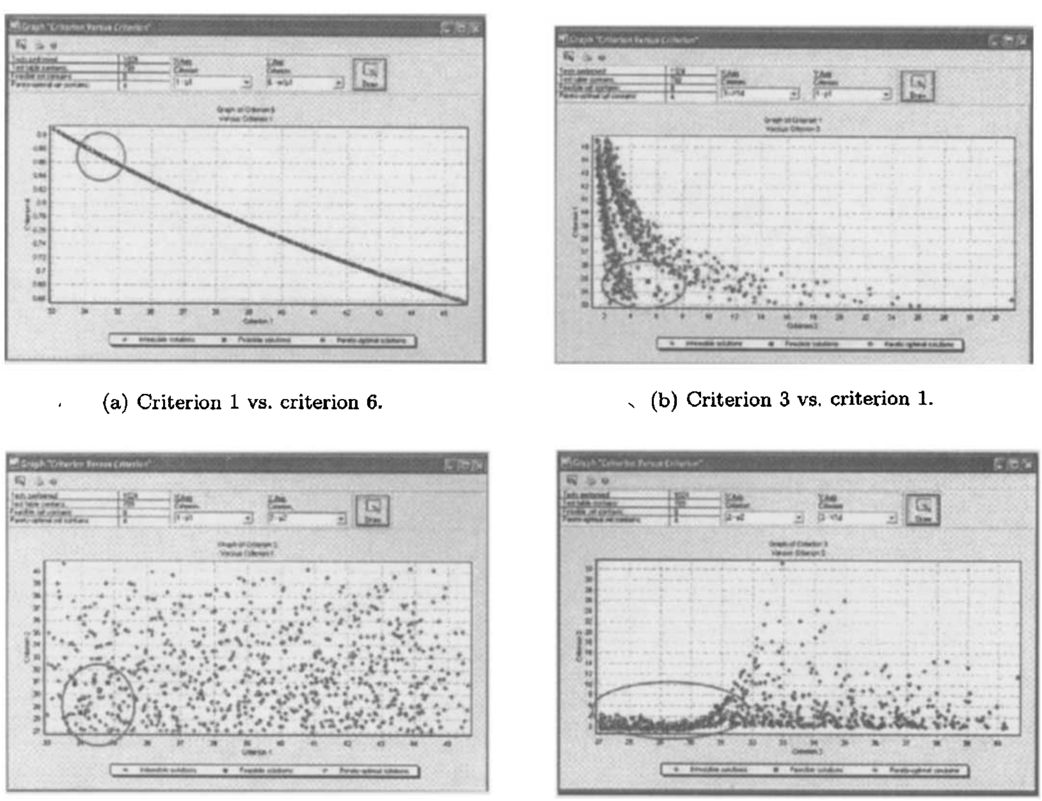
Figure 7. The dependencies of criteria on the first design variable. The regions of the feasible and Pareto optimal solutions are circled.

the corresponding subintervals where the feasible solutions belong. In order to improve the obtained feasible solutions, the designer decided to redo the investigation with the modified initial intervals of variation of design variables α_1, α_3, and α_4 (as shown in the last column in Table 1) and to keep the initial intervals for α_2 and α_5 (i.e., as in (12)). This defines a new parallelepiped Π_1, which was used for *Experiment 2*.

As we also have mentioned, in addition to histograms the designer obtains information in the form of tables containing values of feasible and Pareto optimal vectors of criteria and design variables. Eight feasible solutions are given in Figure 6, four Pareto optimal solutions of which were shown above. Since pseudocriteria are not taken into consideration when constructing Pareto optimal solutions, only the criteria values are presented in Figure 6. Based on an analysis of the Pareto optimal solutions, the designer chooses the most preferable solution.

It is also important to analyze the influence of design variables on criteria. For example, Figure 7 shows the dependencies of criteria Φ_1, Φ_2, Φ_3, and Φ_6 on design variable α_1. We can conclude from Figure 7 that criteria Φ_1 and Φ_6 are antagonistic with respect to α_1. This means that further improvement of criterion Φ_1 is possible by decreasing the value of α_1^*, which results in a deterioration of the value of Φ_6. The criterion Φ_3 is also dependent on α_1, while the dependency of Φ_2 on α_1 is not obvious. These figures were produced in MOVI using the option *graphs criterion vs. design variable II*.

In order to make decisions about the most preferable solution in Pareto set, it is necessary to analyze the dependencies between criteria that are shown in Figure 8. We can see the antagonism of the first and sixth criteria and the rather complex relationships between the remaining criteria. These figures were produced in MOVI using the option *graphs criterion vs. criterion*.



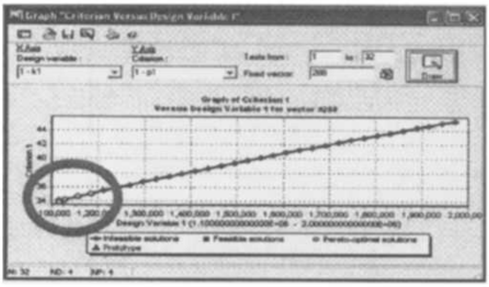
(a) Criterion 1 vs. criterion 6.

(b) Criterion 3 vs. criterion 1.

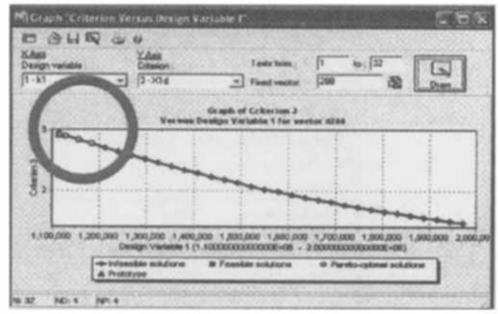
(c) Criterion 1 vs. criterion 2.

(d) Criterion 2 vs. criterion 3.

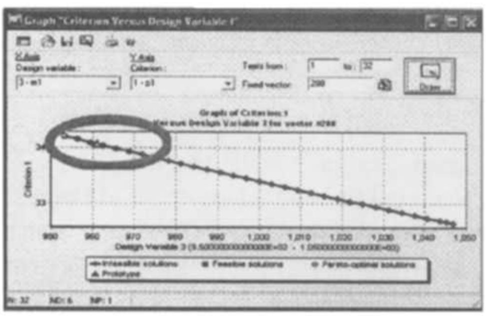
Figure 8. The dependencies between criteria. The regions of the feasible and Pareto optimal solutions are circled.



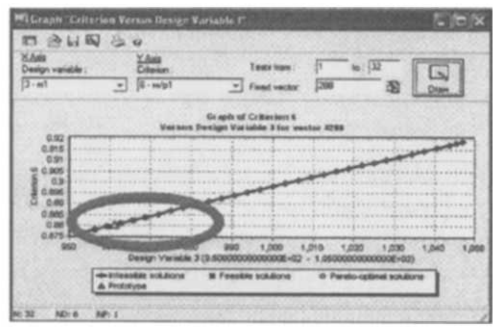
(a) Criterion 1 vs. design variable 1.



(b) Criterion 3 vs. design variable 1.



(c) Criterion 1 vs. design variable 3.



(d) Criterion 6 vs. design variable 3.

Figure 9. The dependency of a criterion on design variables for Pareto optimal solution #288. The regions of the feasible and Pareto optimal solutions are circled.

Suppose that after analyzing the Pareto optimal solutions, the designer gives preference to criterion vector #288. Figure 9 shows the dependencies of criteria on design variables for vector #288 (when one design variable is changing while all the remaining design variables are fixed to Pareto optimal). We can see that criteria Φ_1 and Φ_3 are antagonistic with respect to α_1. Similarly, criteria Φ_1 and Φ_6 are antagonistic with respect to α_3. These figures were produced in MOVI using the option *graphs criterion vs. design variable I*.

EXPERIMENT 2. IMPROVING THE FEASIBLE SOLUTION SET BY CHANGING THE INITIAL INTERVALS OF VARIATION OF THE DESIGN VARIABLES. In this experiment we are seeking to improve the feasible solution set obtained in *Experiment 1* by using a new parallelepiped Π_1. Functional and criteria constraints were the same in both experiments. After 1024 tests using LP_τ sequences, the number of feasible solutions is 258 (compared to eight in *Experiment 1*), and the number of Pareto optimal solutions is 25 (compared to four in the previous experiment). Next, we combined feasible solution sets from both experiments and determined Pareto optimal solutions on the combined feasible solution set. The combined Pareto optimal set contains only 25 solutions, and all of them were obtained in *Experiment 2*. Thus, all solutions from *Experiment 1* were improved.

EXPERIMENT 3. IMPROVING THE FEASIBLE SOLUTION SET BY CORRECTING FUNCTIONAL CONSTRAINTS. As it has already been mentioned, owing to the difficulty of determining functional constraints, the feasible set is often determined incorrectly in applied optimization problems and the search for optimal solutions often loses any practical meaning.

In *Experiment 2* after 1024 trails, 419 did not satisfy the functional constraints: 333 solutions in the second and 86 solutions in the third. All solutions satisfied the first constraint. See Figure 10. Figure 10 is a table of functional failures in the third functional constraint. As indicated in Figure 10 the relation $f_3(\alpha^i) < 27$ holds for all 86 vectors. Only 9 of the 86 vectors are shown.

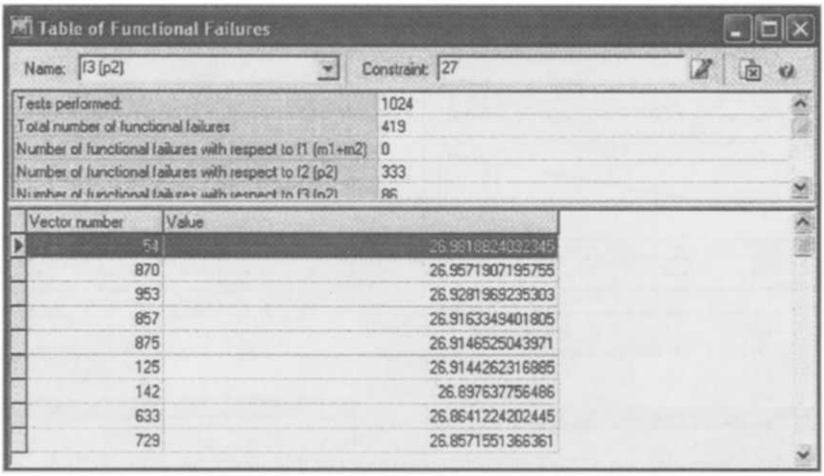


Table of Functional Failures

Name: f3 (p2) Constraint: 27

Tests performed	1024
Total number of functional failures	419
Number of functional failures with respect to f1 (m1+m2)	0
Number of functional failures with respect to f2 (p2)	333
Number of functional failures with respect to f3 (p2)	86

Vector number	Value
54	26.9818824032345
870	26.9571907195755
953	26.9281969235303
857	26.9163349401805
875	26.9146525043971
125	26.9144262316885
142	26.897637756486
633	26.8641224202445
729	26.8571551366361

Figure 10. Table of functional failures.

It follows from Figure 10 that if in place of the given constraint equal to 27, we had made a small concession to a value of 26.85, these nine vectors would not only have satisfied the relaxed constraint, but would also have entered the test table, since this constraint is the last. (Note that the constraints are verified consecutively.) Figure 10 was produced in MOVI using the option *tables of functional failures*.

An analysis of the tables of functional failures allowed the designer in *Experiment 3* to make relatively small concessions from the initial values $33 \le f_2(\alpha)$ and $27 \le f_3(\alpha)$ to $32.5 \le f_2(\alpha)$ and $26.5 \le f_3(\alpha)$. As compared with *Experiment 2* another 24 vectors entered the test table. Thus, with a relatively small relaxation of the initial functional constraints in *Experiment 2*, the number of feasible solutions in *Experiment 3* increased from 258 to 282, and Pareto optimal solutions from 25 to 26. Note that in *Experiment 3*, there is no need to carry out additional trials. The 24 vectors that were found were obtained solely by relaxing the indicated functional constraints.

To summarize, an analysis of the results obtained in *Experiment 1* showed the advisability of correcting the design variable constraints. As a result, in *Experiment 2*, the number of feasible and Pareto optimal solutions were significantly increased from 8 and 4 to 258 and 25, respectively. None of the solutions found in *Experiment 1* entered the combined Pareto set; i.e., all these solutions were improved. *Experiment 3* showed that it was possible to increase the number of feasible and Pareto optimal solutions by correcting the functional constraints. These numbers in comparison with *Experiment 2* increased to 282 and 26, respectively.

REMARKS.

- One measure of improving the statement of a problem is an increase in the efficiency coefficient γ. The quantity γ may be defined as the ratio of the number of feasible solution to the number of trials. Thus, in *Experiment 1*, $\gamma = 8/1024 = 0.0078$. In *Experiment 2*, it increased to $\gamma = 258/1024 = 0.252$, and in *Experiment 3*, the coefficient increased even more to $\gamma = 282/1024 = 0.2754$.
- *Comment on the designer's behavior*. The designer makes a decision about modifying the initial statement of the problem after analyzing the obtained results, i.e., how much the main performance criteria have been improved.

7. ANALYZING COMPUTATIONALLY EXPENSIVE PROBLEMS

For many applied optimization problems, it is necessary to carry out a large-scale numerical experiment in order to construct the feasible set. For this reason, a search for optimal solutions is often not carried out at all. We will mention a few types of difficult problems.

The first type: problems with stringent constraints, as a result of which we obtain small values of γ, for example, $\gamma \ll 0.001$. (Recall that γ is the ratio of the number of feasible solutions to the number of trials.) In this case, even if the time for calculating one criteria vector is fairly short, it takes a long time to find at least one feasible solution because of the need to carry out a large number of trials. These problems are said to be like "looking for a needle in a haystack".

The second type: problems with a high dimensionality of the design variable vectors (e.g., thousands of design variables). It is obvious that these problems also require a large-scale numerical experiment with hundreds of thousands or millions of trials.

The third type: problems with complex mathematical models, where calculating one criteria vector requires a lot of computer time, i.e., from ten or more minutes to many hours. For example, this includes many problems with finite element models.

Below we consider two approaches to solving these problems.

7.1. Parameter Space Investigation in Parallel Mode

The software package MOVI allows one to tackle computationally expensive problems in *parallel mode*, so that the desired number of trials N is distributed among k computers [21]. Thus, each computer finds a feasible solution set for its own subproblem (by conducting $\sim N/k$ trials). Next, all feasible solution sets are combined and a single Pareto optimal solution set is constructed.

EXAMPLE 1. Consider a system with 1000 design variables. The design variable vector is given by $\alpha = (\alpha_1, \ldots, \alpha_{1000})$, $1 \le \alpha_i \le 2$, $i = 1, \ldots, 1000$. We are seeking to minimize simultaneously the following performance criteria $\Phi_v(\alpha)$:

$$\begin{aligned}
\Phi_1 &= \sum_{1}^{1000} \alpha_i, \\
\Phi_2 &= \sum_{300}^{1000} \alpha_i^2 - \sum_{1}^{299} \alpha_i^2, \\
\Phi_3 &= \left(\frac{1400}{\sum_{300}^{1000} \alpha_i^2} \right) - \cos\left(\sum_{i}^{299} \alpha_i \right), \\
\Phi_4 &= \sum_{1}^{700} \frac{\alpha_i}{i} - \left(\sin\left(\sum_{701}^{1000} \alpha_i^2 \right) \right)^5 .
\end{aligned} \tag{16}$$

While analyzing the test tables, we formulated the following criteria constraints:

$$\begin{aligned}
\Phi_1 &< 1502.2254, \\
\Phi_2 &< 930.4528, \\
\Phi_3 &< 0.1624, \\
\Phi_4 &< 10.3851.
\end{aligned} \tag{17}$$

We investigated the parameter (design variable) space and criteria space using four computers simultaneously. Each computer conducted 50,000 trials using a random number generator. The four computers conducted a total of 200,000 trials, which resulted in 4297 feasible solutions, $\gamma = 4297/200000 \approx 0.02$. The CPU time was approximately eight hours per computer using Intel Xeon 2.4 GHz, 2 GB RAM workstations.

After we combined all 4297 feasible solutions, we obtained 326 Pareto optimal ones. The efficiency coefficients for the Pareto optimal and feasible solutions are equal to $\gamma_p = 0.0016$ and $\gamma_f = 0.021$, respectively.

EXAMPLE 2. The following performance criteria need to be minimized:

$$
\begin{aligned}
\Phi_1 &= \sum_{1}^{50} \alpha_i, \\
\Phi_2 &= \sum_{21}^{50} \alpha_i^2 - \sum_{1}^{20} \alpha_i^2, \\
\Phi_3 &= \left(\frac{1400}{\sum_{21}^{50} \alpha_i^2} \right), \\
\Phi_4 &= \sum_{15}^{25} \frac{\alpha_i}{i} - \left(\sum_{26}^{50} \alpha_i^2 \right)^3 .
\end{aligned}
\tag{18}
$$

We have 50 design variables with the following intervals of variation: $1 \le \alpha_i \le 2$, $i = 1, \dots, 50$. We are also given *a priori* criteria constraints

$$
\begin{aligned}
\Phi_1^{**} &= 69.804, \\
\Phi_2^{**} &= 20.384, \\
\Phi_3^{**} &= 23.570, \\
\Phi_4^{**} &= -120,600.
\end{aligned}
$$

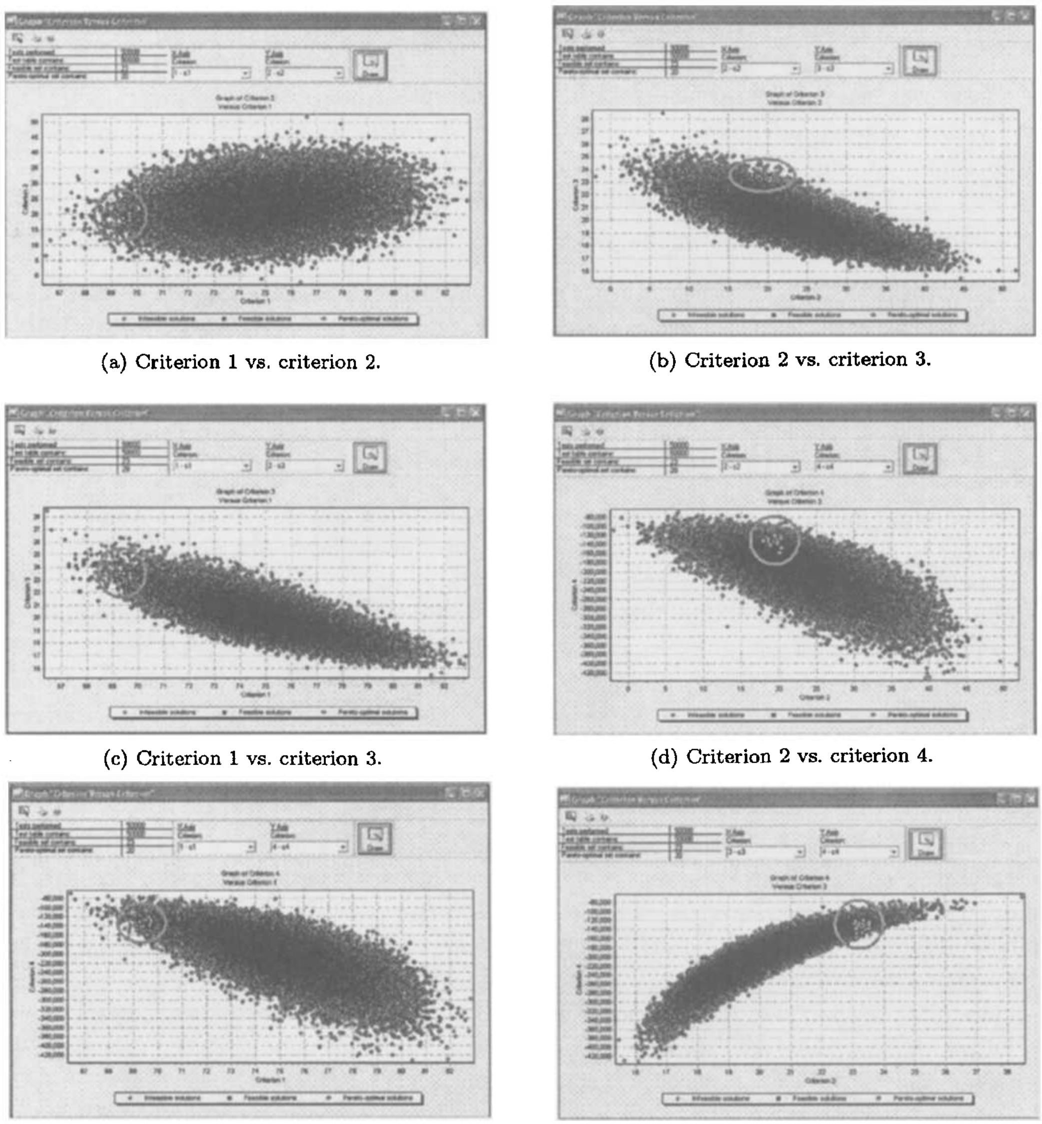
A total of 250,000 trials was conducted on five computers (50,000 trials each) using a random number generator. The combined feasible solution set was constructed, and the combined Pareto optimal set was constructed on it.

Table 2. Pareto optimal solutions obtained on five computers.

Number of a Computer	Feasible and Pareto Optimal Solutions	The Contribution of Each Computer to the Combined Pareto Optimal Solution Set
1	23 (20)	14
2	14 (11)	9
3	13 (10)	8
4	19 (19)	14
5	18 (13)	12

The results of the investigation are presented in Table 2. The combined feasible set contains 87 solutions, and the combined Pareto optimal set contains 57 solutions. For example, data from the first computer are given in the first row: 23 feasible solutions, 20 of which are Pareto optimal solutions; the first computer contributes 14 vectors to the combined Pareto optimal solution set. The contribution of each computer to the combined Pareto optimal solution set is shown in the last column. The coefficients for the Pareto optimal and feasible solutions are equal to $\gamma_p = 0.000228$ and $\gamma_f = 0.000348$, respectively.

The dependencies between criteria obtained on the first computer after carrying out 50,000 trials are shown in Figure 11. This analysis shows the complex relationships between the criteria and the localization of the feasible solutions.



(a) Criterion 1 vs. criterion 2.

(b) Criterion 2 vs. criterion 3.

(c) Criterion 1 vs. criterion 3.

(d) Criterion 2 vs. criterion 4.

(e) Criterion 1 vs. criterion 4.

(f) Criterion 3 vs. criterion 4.

Figure 11. Dependencies between criteria. The regions of the feasible and Pareto optimal solutions are circled.

7.2. Approximating a True Mathematical Model

In Section 2, we defined the solutions satisfying all constraints as feasible solutions. In calculating them, we turned to a mathematical model that we assume to be true. If we replace the true model with an approximate one, we consider the solutions obtained using this model to be *approximate feasible* and *approximate Pareto optimal solutions*. The essence of the approach under consideration is as follows:

- A large number of trials is conducted using the true model in the PSI method. The solutions $\Phi(\alpha^i)$ that entered the test tables are determined.
- An approximate mathematical model is constructed using machine learning algorithms (e.g., [22]), and the approximate feasible solutions $\Phi^p(\alpha^{i_p})$ (satisfying all constraints) are determined by means of the PSI method.
- The obtained approximate feasible solution set is checked for feasibility. To do this, we turn to the true model and calculate the vectors $\Phi(\alpha^{i_p})$ for each α^{i_p}.

The effectiveness of using an approximate model may be judged by the following:

(1) time required to obtain one feasible solution using an approximate and a true model (an approximate model may work much faster than the true model);
(2) the number of references to the true model to check the feasibility of approximate feasible solutions (calculation of the values $\Phi(\alpha^{i_p})$);
(3) statistical estimates of the quality of the approximate model and the obtained solutions.

In Section 8.4, we give an example of obtaining solutions using an approximate model.

8. OTHER CLASSES OF PROBLEMS AND THEIR SOLUTIONS

Solution of the problems described below is based on the multicriteria design method discussed in Section 6.

8.1. Multicriteria Optimal Design of Controlled Engineering Systems

The operating efficiency of the majority of complex engineering systems (automobiles, airplanes, or their engines) strongly depends on the perfection of the system design and the quality of control in specific operating conditions.

The traditional approach to creating controlled engineering systems involves the solution of two optimization problems: the optimal design problem and the optimal control problem. These problems are solved successively and independently of each other. As a rule, the requirements for the efficiency of the automatic control system are not taken into account at the design stage. This philosophy is reflected even in the structure of organizations involved in the development of complex engineering systems; in such organizations, design and control problems are solved in different departments.

In this context, the designer determining optimal control laws has to deal with rigidly fixed structural variables (design variables) of the object, which substantially reduces the possibilities of improving the object's operating efficiency. In fact, the results of the optimal design serve as input data for solving optimal control problems and therefore play a determining role for both the control itself and the efficiency of the entire system.

Thus, it is reasonable to combine the optimal design problem with the optimal control problem to form a single *problem of optimal design of controlled systems*. The proposed solution of this problem would involve simultaneous optimization of design variables and control laws.

Consider an engineering system whose efficiency can be evaluated by a number of *particular performance criteria* Φ_v, $v = 1, \dots, k$. It is important that the set of criteria Φ_v comprise both *'pure design' criteria* Φ_{dv}, $v = 1, \dots, k_1$ and *control criteria* Φ_{cv}, $v = k_1 + 1, \dots, k$. The design criteria can be the mass of the system, the stiffness of the structure, stability margins, the efficiency of the system operation in various operating modes, and so on. Some of the control criteria may coincide with design criteria (e.g., the efficiency of the system operation), while the other control criteria may evaluate specific control characteristics, such as the transition time between operating modes, control stability, energy consumption for control, etc.

The efficiency of this engineering system is determined by a set of *design variables* (structural parameters) $\alpha_d = (\alpha_{d1}, \dots, \alpha_{dp})$ and a set of control laws $u = (u_1, \dots, u_z)$, where p is the number of design variables and z is the number of controlled elements. In general, control laws are functions of time and the variables w_i, $i = 1, \dots, q$ characterizing the *operating mode* of the system, so that $u = f(t, w)$. The number of *'mode variables'* w_i and their physical sense are specific for each engineering system. For example, the mode variables of a gas turbine aircraft engine are the position of the engine control lever; reduced rotation rates n_r of the rotors; pressure, temperature, and humidity of the atmospheric air; and the Mach number. We represent the control vector by a set of *control variables* $\alpha_c = (\alpha_{c1}, \dots, \alpha_{cm})$. For example, these variables can be the coefficients of the function $u = f(t, w)$. We emphasize that any particular performance criterion of an engineering system can be represented as a function of the design variable vector α_d and the control variable vector α_c, so that $\Phi_v = \Phi_v(\alpha_d, \alpha_c)$.

The traditional approach to optimizing controlled engineering systems results in the determination of a single design variable vector α_d (the design of the system) and a corresponding single control variable vector α_c (the set of control laws). This approach does not always make it possible to investigate all potentials for increasing the efficiency of a control system.

A more efficient approach to the multicriteria optimization of controlled engineering systems employs, for a single design variable vector, not a single control variable vector, but a set of vectors, each of which determines the optimal set of control laws for each purpose (operating mode). All the control variable vectors are stored in a computer memory and may be chosen in accordance with a specific control purpose, thus implementing the optimal control. When using this approach, one has, first of all, to construct a set $\tilde{D}$ of feasible solutions $\alpha = (\alpha_d, \alpha_{ci}) \in \tilde{D}$, $i = 1, \ldots, p_\alpha$, where p_α are sets of control laws (specified by the control variable vectors α_{ci}) that correspond to each design (specified by the design variable vector α_d). Then it is necessary to determine a set $\tilde{P} \subseteq \tilde{D}$ of Pareto optimal designs and to select from this set a design $\alpha^0 = (\alpha_d^0, \alpha_{ci}^0) \in \tilde{P}$, $i = 1, \ldots, p_{\alpha^0}$ that is most preferable from the viewpoint of the designer.

In typical multidimensional problems, the number of design variables and control variables may reach many dozens, and thus it is extremely difficult to construct the feasible set $\tilde{D}$. For this reason, we suggest solving practical problems in stages as follows.

STAGE 1. Determine the feasible set D consisting of the design and control variable vectors, $\alpha = (\alpha_d, \alpha_c)$. As a result of this stage, only one set of control laws (represented by the control variable vector α_c) corresponds to each feasible design α_d.

STAGE 2. To estimate the limiting performance of the system, one must solve the multicriteria problem of optimizing the control variables with respect to the control criteria Φ_{cv}, $v = k_1 + 1, \ldots, k$ for all feasible designs. In other words, for each fixed α_d from the set D, by varying only control variables α_c, we construct the vectors $(\alpha_d, \alpha_{ci}) \in \tilde{D}$ in which to any α_d there correspond p_α Pareto optimal control laws. To complete this stage, we determine the set $\tilde{P} \subseteq \tilde{D}$ of Pareto optimal solutions.

STAGE 3. Based on the analysis of the set $\tilde{P}$, select the most preferable solution $\alpha^0 = (\alpha_d^0, \alpha_{ci}^0)$, $i = 1, \ldots, p_{\alpha^0}$.

If the number of control or/and design variables is large, construction of the set $\tilde{D}$ requires a rather extensive numerical experiment. Conducting such an experiment is sometimes either difficult or even impossible. In this case, in Stage 1, we select from the set $P \subseteq D$ of Pareto optimal solutions a subset of most acceptable vectors $\alpha^j = (\alpha_d^j, \alpha_c^j)$. Then for each of the selected α_d^j we solve the multicriteria control problem in accordance with Stage 2.

The effectiveness of this approach was demonstrated in the search for optimal design variables and control laws for a multistage axial flow compressor of a gas turbine aircraft engine and for a robot [3].

The above strategy also allows one to reach the maximum capabilities of efficiency of complex engineering systems by the choice of most preferable design from the obtained set and by implementation (for example, on an airborne computer) of different control laws optimal for different purposes and operating modes of the engineering system.

8.2. Multicriteria Identification

One of the fundamental problems in engineering optimization is determination of *the adequacy of the mathematical model for the actual object*. Without estimating the model's adequacy, the search for optimal design variables has no applied sense. But what is the measure of adequacy? To what extent can we trust one model or other? In other words, we must ensure that our model is adequate to the system under study [1–4].

We denote by $\Phi_v^c(\alpha)$, $v = 1, \dots, k$, the criteria resulting from the analysis of the mathematical model that describes a physical system, where $\alpha = (\alpha_1, \dots, \alpha_r)$ is the vector of the parameters of the model. The criteria $\Phi_v^c(\alpha)$ can be functionals of integral curves of differential equations or functions of the vector α that are not associated with solutions of differential equations.

Let Φ_v^{exp} be the experimental value of the v^{th} criterion measured directly on the prototype. Suppose there is a mathematical model or a hierarchical set of models describing the system's behavior. Let $\Phi = (\|\Phi_1^c - \Phi_1^{\mathrm{exp}}\|, \dots, \|\Phi_k^c - \Phi_k^{\mathrm{exp}}\|)$, where $\|\cdot\|$ is a particular adequacy (closeness, proximity) criterion. As it has already been mentioned, this criterion is a function of the difference (error) $\Phi_v^c - \Phi_v^{\mathrm{exp}}$. It is often given by $(\Phi_v^c - \Phi_v^{\mathrm{exp}})^2$ or $|\Phi_v^c - \Phi_v^{\mathrm{exp}}|$. If the experimental values Φ_v^{exp}, $v = \overline{1,k}$ are measured with considerable error, then the quantity Φ_v^{exp} can be treated as a random variable. If this random variable is normally distributed, the corresponding adequacy criterion is expressed by $M\{\|\Phi_v^c - \Phi_v^{\mathrm{exp}}\|\}$, where $M\{\|\cdot\|\}$ denotes the mathematical expectation of the random variable $\|\cdot\|$. For other distribution functions, more complicated methods of estimation are used, for example, the maximum likelihood method.

We formulate the following problem by comparing the experimental and calculated data to determine to what extent the model corresponds to the physical system and find the model variables. In other words, it is necessary to find the vectors α^i satisfying conditions (1) and (2) and, in addition, the inequalities

$$\left\|\Phi_v^c\left(\alpha^i\right) - \Phi_v^{\mathrm{exp}}\right\| \le \Phi_v^{**}. \tag{19}$$

Conditions (1), (2), and (19) define the *feasible solution set* D_α. Here, Φ_v^{**} are criteria constraints that are determined in the dialogue between the designer and a computer. To a considerable extent, these constraints depend on the accuracy of the experiment and the physical sense of the criteria Φ_v. Examples of solving identification problems are described in [2–4,8,23].

8.3. Operational Development of Prototypes

The problem of operational development of a prototype and its improvement is one the most pressing and complex design problems. This problem is encountered in the production of machine tools, automobiles, ships, and aircrafts, where enormous amounts of money are spent on the operational development of the object with limited time to solve the problem.

We suggest carrying out the operational development of prototype in two stages. In the *first stage*, accelerated tests (for instance, bench tests) are performed. These tests allow us to identify the mathematical model of the object and to determine its parameters. Thus, the set D_α is found as a result of multicriteria identification. In the *second stage*, the designer formulates and solves the multicriteria optimization problem. We construct the parallelepiped Π in D_α, determine the vector of performance criteria, and find the feasible solution set D. To do this, we use the mathematical model whose adequacy was established in the first stage. Based on the optimization results, improvements to the prototype are made, and then the bench tests and full-scale test are conducted. This cycle is repeated until the designer decides to terminate the operational development.

Let us summarize the characteristic features of these problems:

- The designer has insufficient information about design variable constraints before solving identification problems.
- The presence of strong design variable, functional, and criteria constraints (the object already exists and we need to update it).
- High dimensionality of criteria vector. For complex systems, the number of particular proximity criteria used to evaluate the adequacy of the mathematical model can reach many dozens, e.g., a 65-criteria identification problem of *operational development of a vehicle* was solved by application of the PSI method and is described in [2].

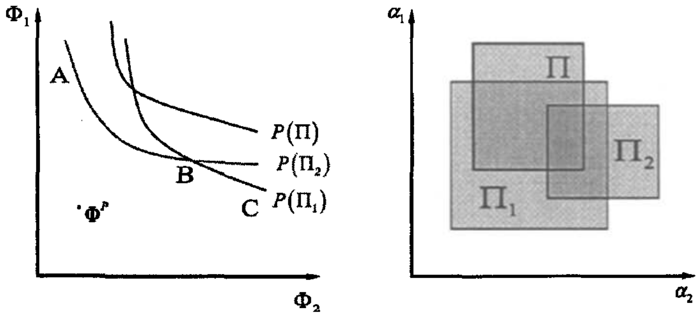


Figure 12. Approaches to solving problems of improving a prototype.

Some ways of solving problems of operational development of prototypes are considered below.

Suppose we have a prototype that needs to be improved. The criteria vector of the prototype is known, $\Phi^p = (\Phi_1^p, \dots, \Phi_k^p)$. Figure 12a shows the criteria space of the prototype vector Φ^p, while Figure 12b shows the design variable space of the initial parallelepiped Π.

It is desirable to find the design variable vector α^i that satisfies the inequalities

$$\left\|\Phi_v\left(\alpha^i\right) - \Phi_v^p\right\| \le \Phi_v^{**}, \qquad v = 1, \dots, k,$$

and, on the set of Pareto optimal vectors, determine the most preferred solution $\Phi(\alpha^o)$, surpassing the prototype in all criteria or at least the most important ones. Let us consider two situations:

1. A solution Φ^p exists. However since the designer has only a rough idea of the possible search limits for many of the identified design variables, the identified vector is usually $\alpha^p \notin \Pi$. In view of this, the initial parallelepiped Π, and in a number of cases, the mathematical model itself, must be repeatedly corrected.
2. Equally important is the situation where it is impossible to identify the vector α^p, for example, when the designer's wishes for local criteria Φ_v^p, $v = 1, \dots, k$ are unattainable. Here, the search process of the prototype is very useful, since it allows one to define the compromise solution $\Phi(\alpha^o)$ that in a sense is close to $\Phi^p = (\Phi_1^p, \dots, \Phi_k^p)$, if not in all local criteria Φ_v^p, then at least in the most important ones. Thus, we can answer the question of how to improve the prototype and by how much.

Suppose a Pareto optimal solution set $P(\Pi)$ is constructed given some initial constraints, but the designer is not satisfied with the obtained solutions, Figure 12a. Based on an analysis of the results in Π, the statement of the problem is corrected, for example, the design variable constraints, and a new parallelepiped Π_1 is constructed. Figure 12b shows Π_1, while Figure 12a shows the Pareto optimal set $P(\Pi_1)$ corresponding to it. Figure 12a also shows Pareto optimal sets $P(\Pi)$, $P(\Pi_1)$, and $P(\Pi_2)$. The region of best approximations to Φ^p obtained as a result of investigating Π, Π_1, and Π_2 consists of the curves AB and BC: AB belongs to $P(\Pi_1)$ and BC belongs to $P(\Pi_2)$. The solution of similar problems includes correcting all restraints according to the results of an investigation of the criteria space and design variable space.

EXAMPLE 1. We will consider the problem of improving a prototype using the example of a dynamic system (11).

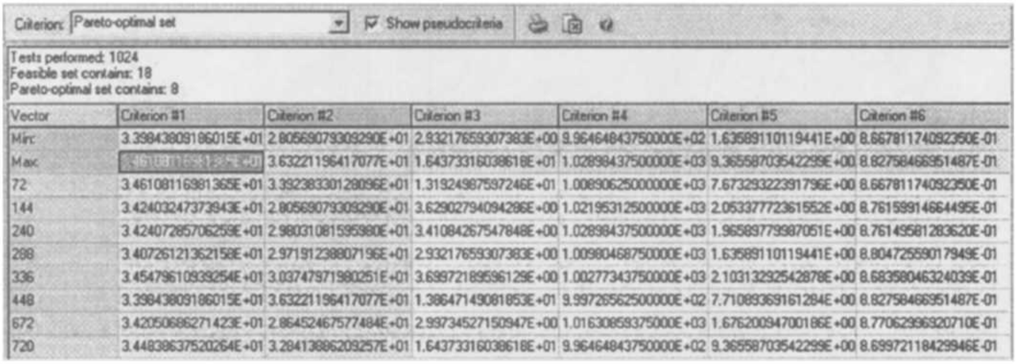
Let us consider investigation in parallelepiped Π, see (12). The criteria vector of the prototype is given

$$\Phi^p = (32.616; 41.231; 20.633; 970; 10.316; 0.91978).$$

Recall that the first two indices $\Phi_1^p = 32.616$ and $\Phi_2^p = 41.231$ are pseudocriteria, and the remaining four Φ_3^p, Φ_4^p, Φ_5^p, Φ_6^p are criteria. The boundaries of the initial parallelepiped are defined in (12). *A priori* criteria constraints are stated (some of them are larger than the values of the criteria Φ^p), in particular,

$$\Phi^{**} = (34.616; 42.231; 20.733; 1030; 10.416; 0.92078);$$

in addition, for the prototype, the requirements $f_1 \le 1100$, $f_2 \ge 33$, and $f_3 \ge 24$ must be fulfilled.

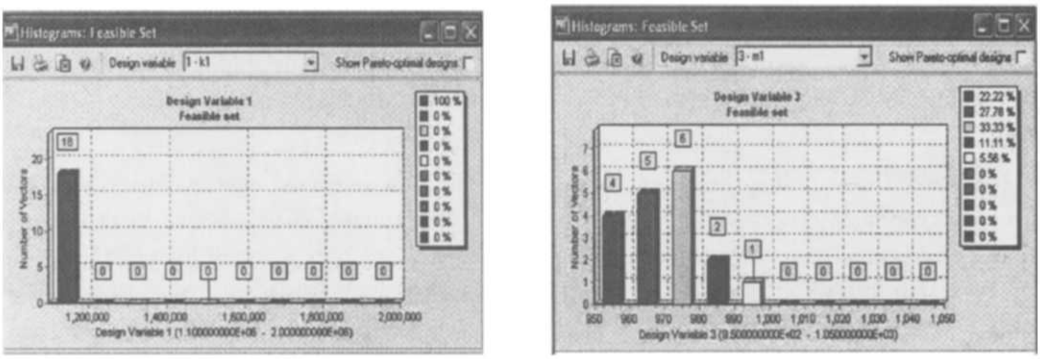


Criterion: Pareto-optimal set — Show pseudocriteria

Tests performed: 1024
Feasible set contains: 18
Pareto-optimal set contains: 8

Vector	Criterion #1	Criterion #2	Criterion #3	Criterion #4	Criterion #5	Criterion #6
Min:	3.39843809186015E+01	2.80569079309290E+01	2.93217659307383E+00	9.96464843750000E+02	1.63589110119441E+00	8.66781174092350E-01
Max:	[illegible]	3.63221196417077E+01	1.64373316038618E+01	1.02898437500000E+03	9.36558703542299E+00	8.82758466951487E-01
72	3.46108116981365E+01	3.39238330128096E+01	1.31924987597246E+01	1.00890625000000E+03	7.67329322391796E+00	8.66781174092350E-01
144	3.42403247373943E+01	2.80569079309290E+01	3.62902794094286E+00	1.02195312500000E+03	2.05337772361552E+00	8.76159914664495E-01
240	3.42407285706259E+01	2.98031081595980E+01	3.41084267547848E+00	1.02898437500000E+03	1.96589779987051E+00	8.76149581283620E-01
288	3.40726121362158E+01	2.97191238807196E+01	2.93217659307383E+00	1.00980468750000E+03	1.63589110119441E+00	8.80472559017949E-01
336	3.45479610939254E+01	3.03747971980251E+01	3.69972189596129E+00	1.00277343750000E+03	2.10313292542878E+00	8.68358046324039E-01
448	3.39843809186015E+01	3.63221196417077E+01	1.38647149081853E+01	9.99726562500000E+02	7.71089369161284E+00	8.82758466951487E-01
672	3.42050686271423E+01	2.86452467577484E+01	2.99734527150947E+00	1.01630859375000E+03	1.67620094700186E+00	8.77062996920710E-01
720	3.44838637520264E+01	3.28413886209257E+01	1.64373316038618E+01	9.96464843750000E+02	9.36558703542299E+00	8.69972118429946E-01

Figure 13. Pareto optimal solutions (criteria vectors) in parallelpiped Π.



(a) First design variable. (b) Third design variable.

Figure 14. Histograms of the distribution of feasible solutions in parallelepiped Π.

A total of 1024 trials was conducted in Π, and 18 feasible solutions were found, of which eight were Pareto optimal, see Figure 13. In Figure 13, criterion 1 and criterion 2 are pseudocriteria. Analysis of the obtained results showed that vectors #720, #448, and #72 are quite close to the prototype in the criteria. They are slightly inferior to it in the fourth criterion and surpass it in the third, fifth, and sixth criteria. As a result of analyzing the boundaries of the design variables, the designer makes a decision on further investigations by decreasing the lower intervals of variation of the first and third design variables. The advisability of this can be seen from the histograms of the distribution of feasible solutions (see Figure 14). Based on this, a new parallelepiped Π_1 is constructed, see Table 3.

Table 3. Boundaries of the variable parameters in the two experiments.

	α_1	α_2	α_3	α_4	α_5
Π	$[1.1 \cdot 10^6; 2.0 \cdot 10^6]$	$[4.0 \cdot 10^4]$	$[950; 1050]$	$[30; 70]$	$[80; 120]$
Π_1	$[0.9 \cdot 10^6; 1.2 \cdot 10^6]$	$[3.5 \cdot 10^4; 5.50 \cdot 10^4]$	$[900; 1010]$	$[25; 65]$	$[70; 130]$

Now we perform investigation in parallelepiped Π_1. A total of 1024 trials was conducted with the criteria constraints given above. 110 feasible solutions were found, 13 of which were Pareto optimal, where eight of these (#101, #793, #441, #126, #854, #378, #390, #882) surpassed the prototype Φ^p in all four criteria, see Figure 15. Analysis of the feasible values of the design variables and the histograms showed the possibility of a further search for the best solutions by correcting the constraints on the first and third design variables.

EXAMPLE 2. THE PROBLEM OF IMPROVING THE PROTOTYPE OF A SHIP. The purpose of this example is to demonstrate multicriteria analysis in six experiments[4] (in four parallelepipeds)

[4] LP_τ sequences were used in Experiments 1–4 and 6, and a random number generator in Experiment 5. We will

Criterion: Pareto-optimal set ☑ Show pseudocriteria

Tests performed: 1024
Feasible set contains: 110
Pareto-optimal set contains: 13

Vector	Criterion #1	Criterion #2	Criterion #3	Criterion #4	Criterion #5	Criterion #6
Min	3.31076502466480E+01	2.78468867523177E+01	1.63997271754716E+00	9.32890625000000E+02	8.54259226112942E-01	8.66952130003350E-01
Max	3.46039867274841E+01	3.97392491221472E+01	1.65040308903491E+01	1.02093750000000E+03	8.18835243539246E+00	9.06134980178408E-01
101	3.46039867274841E+01	2.78468867523177E+01	3.71603487267505E+00	9.66250000000000E+02	2.03365814711631E+00	8.66952130003350E-01
126	3.39712341611295E+01	3.12440470520462E+01	7.84782570627820E+00	9.53750000000000E+02	4.11091182504651E+00	8.83100091615939E-01
158	3.32386116196608E+01	2.99295566976207E+01	1.63997271754716E+00	1.00218750000000E+03	8.54259226112942E-01	9.02564774464131E-01
275	3.45572667549586E+01	2.83633958766731E+01	3.66339163839136E+00	1.00093750000000E+03	2.07890320416916E+00	8.68124212853012E-01
285	3.40474552488868E+01	2.99056389484527E+01	1.73739827410357E+00	1.02093750000000E+03	9.69651478174016E-01	8.81123120089301E-01
378	3.34755878938955E+01	2.78587280174904E+01	4.24286579549021E+00	9.56562500000000E+02	2.14422172770135E+00	8.96175448660924E-01
390	3.34613754667107E+01	3.01069297865462E+01	3.55997459643194E+00	9.47187500000000E+02	1.80536602336632E+00	8.96556091361088E-01
441	3.45422519246720E+01	3.89537071964637E+01	9.59004345913507E+00	9.35937500000000E+02	5.20053821568135E+00	8.68501569191914E-01
590	3.34005293901245E+01	2.95817219201241E+01	2.07159965087205E+00	9.89140625000000E+02	1.07089968280188E+00	8.98189356509723E-01
597	3.41010200776561E+01	2.99093969356153E+01	1.76656240357881E+00	1.00164062500000E+03	9.71178032319032E-01	8.79739079115019E-01
793	3.45538735174428E+01	2.93592096314606E+01	2.05453969207976E+00	9.61015625000000E+02	1.10842817664987E+00	8.68209463834959E-01
854	3.34776929564846E+01	2.98446356147176E+01	4.33324832615904E+00	9.54140625000000E+02	2.22100135154743E+00	8.96119097543398E-01
882	3.31076502466480E+01	3.97392491221472E+01	1.65040308903491E+01	9.32890625000000E+02	8.18835243539246E+00	9.06134980178408E-01

Figure 15. Pareto optimal solutions (criteria vectors) in parallelepiped Π_1.

resulting in an improved prototype, see [23]. We will omit a description of the mathematical model and briefly illustrate some elements of multicriteria analysis. Among the particular features of the problem are the high dimensionality of the design variable vector (45 design variables) and the difficulties of improving a reasonably good prototype under strong constraints (seven functional constraints and nine pseudocriteria). Six criteria were optimized: Φ_1 is the propulsion power factor (%) (min); Φ_2 is the electrical power factor (%) (min); Φ_3 is the volume factor (%) (max); Φ_4 is the region factor (%) (max); Φ_5 is the weight factor (%) (max); and Φ_6 is the cost (min).

In view of the high dimensionality of the design variable vector, 200,000 tests were conducted in each of the first five experiments and 500,000 in the sixth experiment. After each of the first three experiments, the constraints were corrected according to the results of analysis of the test tables, tables of feasible and Pareto optimal solutions, tables of functional failures, histograms of feasible solutions, and graphs of dependencies of criterion versus criterion and criterion versus design variables. Then a new experiment in a new parallelepiped was conducted. The fourth and sixth experiments were conducted in the fourth parallelepiped. Starting from the specified values of the prototype, design variable (parallelepiped Π_1), functional, and criteria constraints were formulated, with the functional and criteria constraints being weakened in comparison with the prototype.

A total of seven feasible solutions (all of them Pareto optimal) was obtained in the *first experiment* (parallelepiped Π_1). No interesting solutions were obtained from the designer's point of view. Based on the results of an analysis of the feasible solutions, the ranges of some of the design variables were corrected and parallelepiped Π_2 was constructed.

The *second experiment* (parallelepiped Π_2) also did not lead to new results. There were nine feasible and three Pareto optimal solutions, respectively. Based on the results of analysis of the second experiment, the design variable constraints were corrected and thus parallelepiped Π_3 was constructed. The functional constraints and criteria constraints were also corrected. These changes formed the essence of the third experiment.

Three feasible (they are also Pareto optimal) solutions were found in the *third experiment* (parallelepiped Π_3): #17311, #108455, and #71279. These solutions attracted the attention of the designer. For example, design #108455 proved to be better than the prototype in five of the six criteria. The smaller number of feasible and Pareto optimal solutions in comparison with the first and second experiments was caused by the considerable strengthened criteria constraints. Based on the results of an analysis of the third experiment, the search region in the fourth experiment

restrict ourselves to describing the experiment with LP_τ sequences.

was limited by the design variable values of the three specified designs. Thus, parallelepiped Π_4 was constructed.

In the *fourth experiment* (parallelepiped Π_4), the criteria constraints were strengthened in comparison with the third experiment (the *first dialogue* of the designer with the computer). However, the number of the feasible and Pareto optimal solutions turned out to be rather high (2161 and 281, respectively). This is due to the fact that the search region in parallelepiped Π_4 was substantially smaller than in parallelepiped Π_3 for the same number of tests. Many solutions of interest to the designer were found. After analyzing the obtained solutions, an attempt was made to improve the prototype in all criteria simultaneously. Therefore, in the *second dialogue*, the criteria constraints corresponded to the values of the prototype criteria. As a result, 20 Pareto optimal solutions surpassing the prototype in all criteria were found. Thus, the problem of improving the prototype has been solved.

Two dialogues were also conducted in the *sixth experiment* (parallelepiped Π_4). In the *first dialogue*, the criteria constraints on the second and sixth criteria were strengthened in comparison with the first dialogue in the fourth experiment. A total of 500,000 trials was conducted, and 627 feasible and 138 Pareto optimal solutions, respectively, were found. Many of them were very interesting for the designer. In the *second dialogue*, the criteria constraints corresponded to the values of the prototype criteria. Eleven Pareto optimal solutions surpassing the prototype in all six criteria simultaneously were found. In comparison with the second dialogue of the fourth experiment, we obtained six new solutions. A combined set of Pareto optimal solutions surpassing the prototype in all six criteria contains 26 solutions, five of which (#16907, #164167, #191033, #293036, #293036, #364925) are given in Table 4.

Table 4 Experimental results.

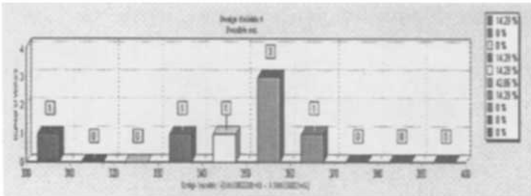
Experiments	Φ_1% (min)	Φ_2% (min)	Φ_3% (max)	Φ_4% (max)	Φ_5% (max)	Φ_6 (min)
Prototype	2.48	10.00	11.77	14.33	1.01	555
Fourth experiment, #16907	2.28	8.10	14.6	18.3	5.68	547
Fourth experiment, #164167	2.22	3.03	19.6	23.7	8.46	549
Fourth experiment, #191033	2.42	7.35	15.1	18.7	8.16	544
Sixth experiment, #293036	2.40	1.55	23.4	27.4	2.10	543
Sixth experiment, #364925	2.37	2.56	24.8	28.8	5.14	547

REMARK. There were similar constraints in the fourth and fifth experiments. As mentioned above, a random number generator was employed in the fifth experiment to investigate the design variable space. A total of 2169 feasible and 184 Pareto optimal solutions was found. The best solutions in the fourth and fifth experiments turned out to be nearly identical.

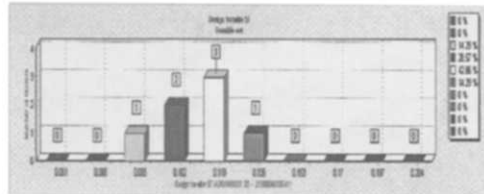
Some elements of the performed multicriteria analysis are shown below. Histograms of the distribution of feasible solutions for the 1^{st} and 37^{th} design variables in the first four experiments are shown in Figure 16. It is clear that a good distribution of feasible solutions was obtained only in the fourth experiment. The dependencies between criteria (third experiment) are shown in Figure 17. The regions of the three specified designs are circled.

In summary, we will draw attention once again to some features and strategy of problem solving.

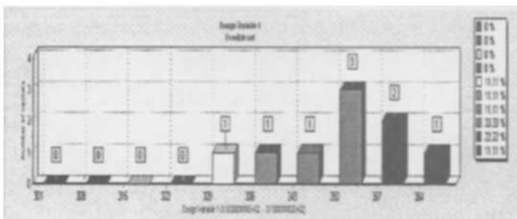
- The dimensionality of the design variable vector was high (equal to 45). Therefore, it was necessary to carry out a large number of trials. A total of 200,000 trials was conducted in each of the first five experiments and 500,000 in the sixth.



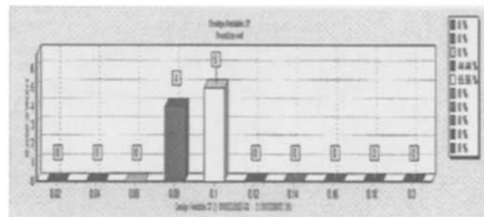
(a) First experiment.



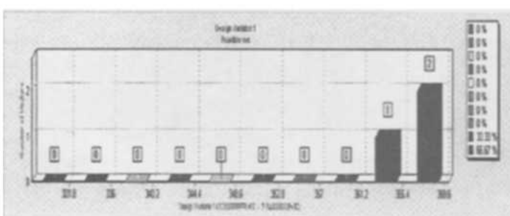
(b) First experiment.



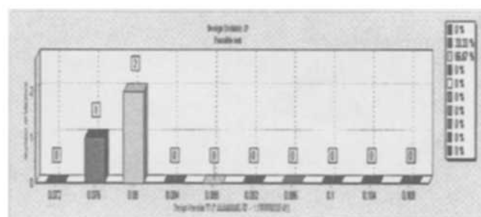
(c) Second experiment.



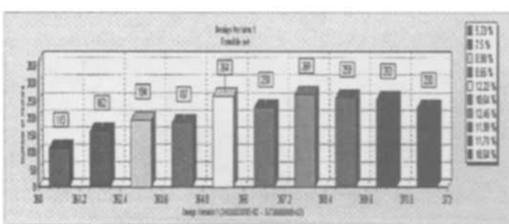
(d) Second experiment.



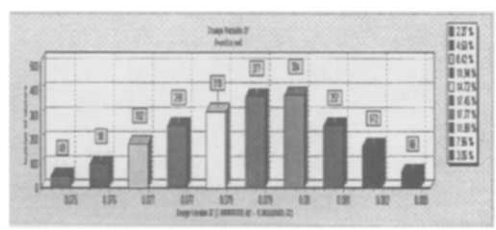
(e) Third experiment.



(f) Third experiment.



(g) Fourth experiment.



(h) Fourth experiment.

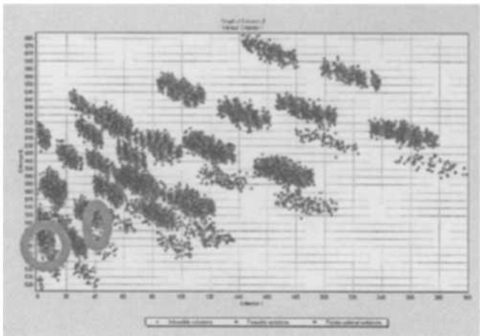
Figure 16. Histograms of the distribution of feasible sets for the 1^{st} and 37^{th} design variables.

- Multicriteria analysis showed the necessity of repeated correction of the constraints, and because of this, a series of experiments was performed. Each subsequent experiment was carried out on the basis of the previous one (step by step). In the first three experiments, we obtained a small number of feasible solutions; and it was only in the third experiment that we came close to satisfactory results. An analysis of these results allowed us to define the region of good solutions where subsequent experiments were carried out.
- Improvement of the prototype in all criteria occurred in the second dialogue of the fourth and sixth experiments.

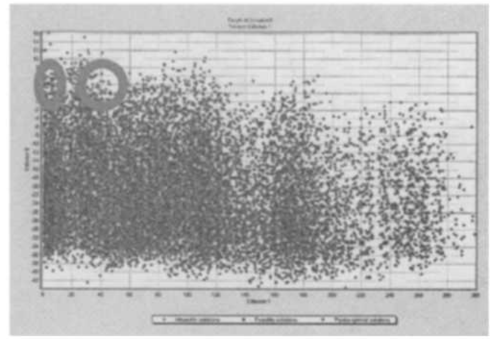
Owing to the difficulties of correctly stating engineering optimization problems, designers end up solving ill-posed problems. By this example, we demonstrated how to state and solve similar problems correctly on the basis of the PSI method.

8.4. Multicriteria Analysis from Observational Data

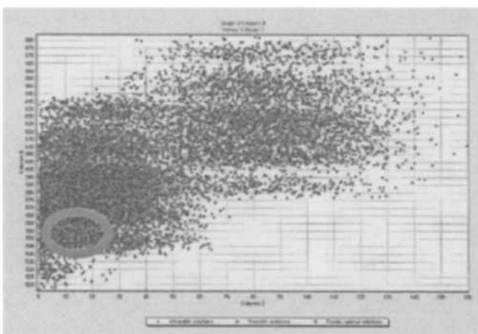
For this class of problems, there are no *a priori* specified mathematical models. However, there are available observations in the form of tables that give an indication of the behavior



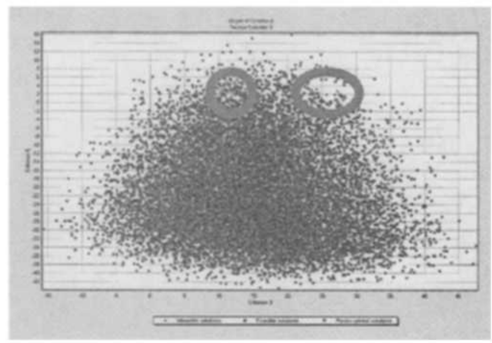
(a) Criterion 6 vs. criterion 1.



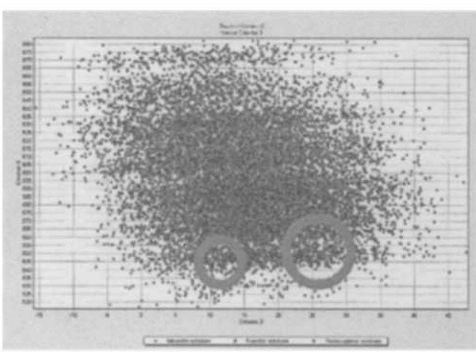
(b) Criterion 5 vs. criterion 1.



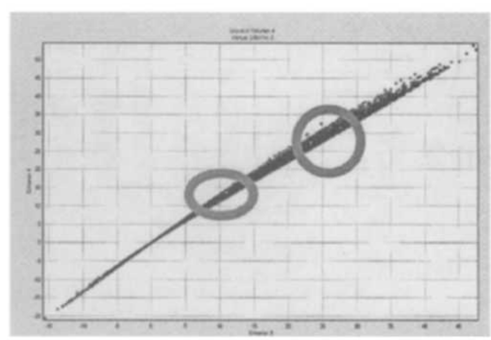
(c) Criterion 6 vs. criterion 2.



(d) Criterion 5 vs. criterion 3.



(e) Criterion 6 vs. criterion 3.



(f) Criterion 4 vs. criterion 3.

Figure 17. Dependencies between criteria in the third experiment. The regions of the feasible and Pareto solutions are circled.

of the system under investigation. These problems are often encountered in medicine, biology, economics, materials science, information science, and other fields. An approximate mathematical model is constructed on the basis of the observations with the use of classification and regression algorithms. Some algorithms for constructing approximate criteria functions include regression by neural networks, support vector machine (SVM) regression, and multiple linear regression [22,24–26]. Below we describe a general strategy for multicriteria analysis from observational data.

STEP 1. OBTAINING OBSERVATIONAL DATA AND CONSTRUCTING AN APPROXIMATE MODEL. Suppose we have an experiment with N observations represented by an $N \times M$ matrix, where M is the total number of observed variables (criteria and design variables). The approximate criteria functions are constructed using machine learning algorithms. The quality of approximate criteria functions is evaluated by statistical metrics, such as R^2 (fraction of variance explained by a model) and absolute, relative, and squared errors. Then functions with the best evaluation performance are chosen for the approximate mathematical model.

STEP 2. MULTICRITERIA ANALYSIS: CONSTRUCTION OF THE APPROXIMATE FEASIBLE SOLUTION SET AND SEARCH FOR THE BEST SOLUTIONS.

EXAMPLE. Below we show the process of constructing the approximate feasible solution set in problems in which only observational data are present. This example also illustrates principles of work with the approximate model described in Section 7.2.

The collection of observational data depends on the specifics of the problem being investigated and is beyond the scope of the present work. In the present case, in order to obtain observational data, we referred to a true model (11) and, using the PSI method, conducted 4000 trials with a random number generator. As a result, we obtained a 4000×11 ($M = 6$ criteria + 5 design variables) matrix of observations.

Using the observational data, we constructed approximate criteria functions by means of machine learning algorithms. In our case, criteria Φ_3 and Φ_5 were determined using generalized neural networks for regression [24], while the remaining four criteria were reconstructed using the SVMTorch algorithm [25]. This choice was based on statistical estimates of the criteria functions obtained; the estimates of the best approximate functions are given in Table 5. These criteria functions constitute the approximate mathematical model.

Table 5. Statistical estimates of the approximate criteria functions.

Criteria	R^2	Mean Absolute Error	Mean Relative Error	Mean Squared Error
Φ_1	0.999997	0.0361611	3.44387e − 005	0.00313826
Φ_2	0.999975	0.0121383	0.000329313	0.000357047
Φ_3	0.810006	0.603323	0.119773	2.35193
Φ_4	0.999997	0.0361611	3.44387e − 005	0.00313826
Φ_5	0.797901	0.368809	0.109251	0.778614
Φ_6	0.998594	0.0024371	0.00319643	7.67688e − 006

At this stage, we have an approximate model and we will use it with the PSI method. That is, we employed the PSI method to conduct 1024 trials using LP_τ sequences. We constructed test tables and obtained eight approximate feasible solutions $\Phi^\nu(\alpha^{i_p})$ that satisfied constraints (12), (14), and (15). These were vectors #288, #336, #520, #544, #560, #672, #896, and #1008.

Since in this example we had access to a true model, the vectors α^{i_p} were checked for feasibility by direct application of the true model and calculation of the values $\Phi(\alpha^{i_p})$. Seven of the eight approximate feasible solutions indicated above were found to be feasible.[5] The eighth approximate feasible solution #1008 was nonfeasible because of errors in the approximate model.

After constructing and analyzing the approximate feasible solution set, we corrected design variable constraints and determined a new approximate feasible set. This procedure was similar to the one described in Section 6.2. After 1024 trials with LP_τ sequences, 311 approximate feasible solutions were identified, 218 of which turned out to be feasible. We note that in *Experiment 2* with the true model, there were 258 feasible solutions.

In order to analyze the efficiency of the employed approximate model, we can use a metric equal to the number of feasible solutions found with the approximate model over the number of feasible solutions obtained with the true model. In the cases described above, this metric is $7/8 = 0.875$ and $218/258 = 0.85$, respectively. Further improvement of the efficiency of an approximate model is possible by improving the fit of the true criteria functions, especially for Φ_3 and Φ_5. However, it is worth noting that we have already approximated the true model fairly

[5]Recall that eight feasible solutions were found from the true model, see *Experiment 1* in Section 6.2.

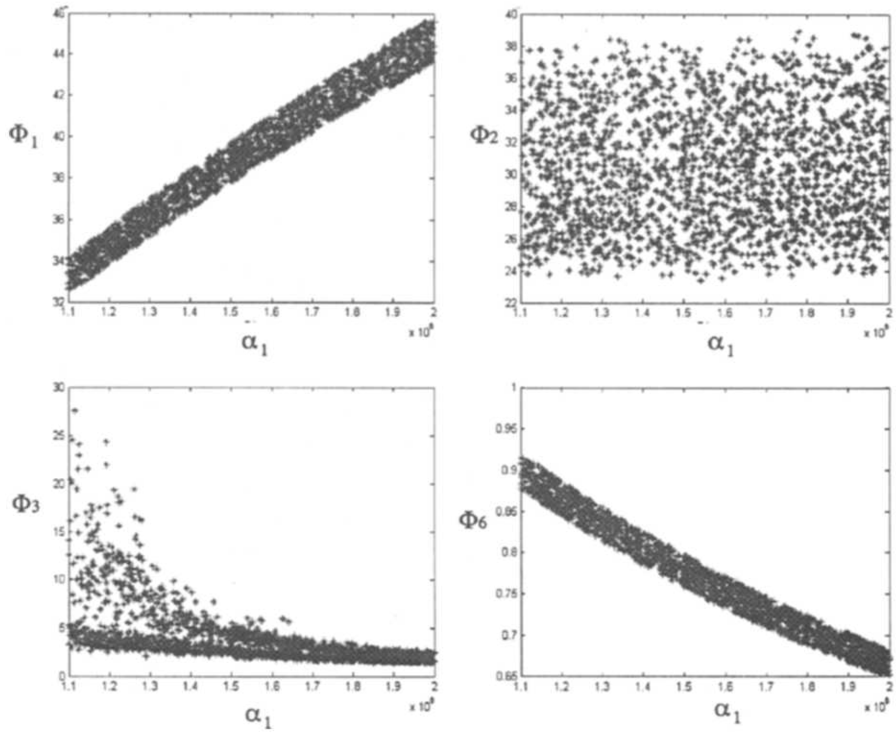


Figure 18. The dependencies of criteria on the first design variable for the approximate mathematical model. See Figure 7 for the true model.

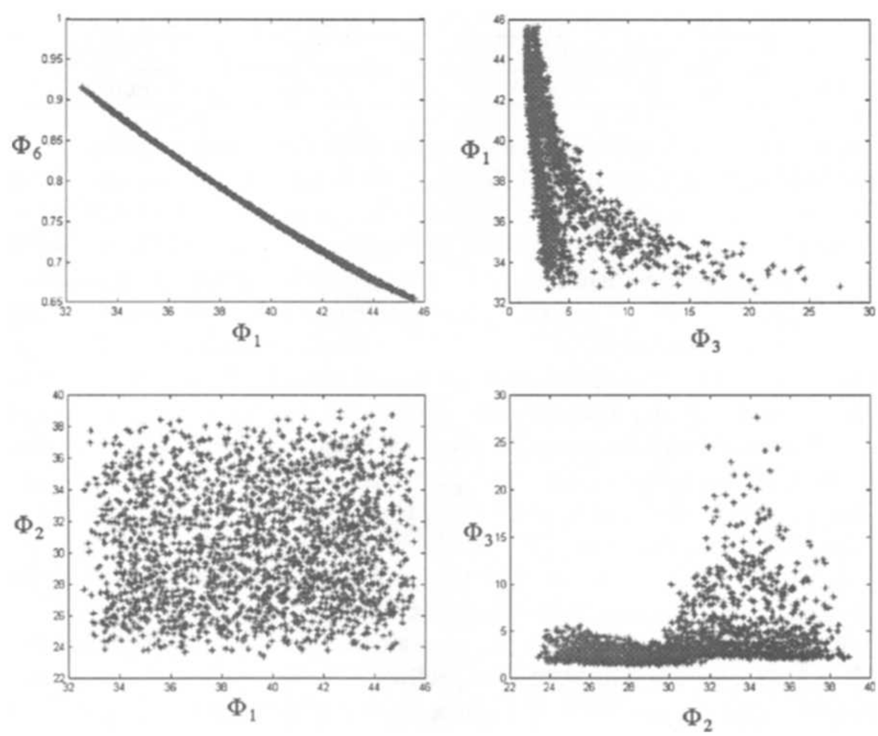


Figure 19. The dependencies between criteria for the approximate mathematical model. See Figure 8 for the true model.

well, such that we have preserved the dependencies of criteria on design variables and between criteria, see Figures 18 and 19.

To summarize, multicriteria analysis can be carried out in problems from observational data by constructing an approximate model. This analysis can be used to predict the best solutions and approaches to their subsequent improvement.

9. CONCLUSIONS

One of the main causes of inefficient use of standard optimization methods for solving applied problems is that it is difficult for the designer to correctly specify the feasible solution set, and therefore, as a rule, one solves ill-posed problems. The problem of constructing a feasible set is a fundamental one and is usually not addressed. The PSI method has been created for the correct definition of the feasible solution set. The MOVI software (implementing the PSI method) is a comprehensive system that enables methodologically rigorous multicriteria analysis. The multicriteria analysis tools available in MOVI, such as:

- test tables,
- tables of feasible and Pareto optimal solutions,
- tables of functional failures,
- histograms of feasible solutions,
- graphs of dependencies of criteria on design variables and dependencies between criteria allow us to:
    1. correctly construct and analyze the feasible and Pareto optimal solution sets,
    2. correct the initial statement of the problem (design variable, functional, and criteria constraints),
    3. make a decision about the most preferable solutions in the Pareto set,
    4. conduct large-scale numerical experiments (with hundreds of thousands or millions of trials),
    5. solve many problems that until recently were impossible to optimize.

Using the PSI method as a basis, it is possible to solve multicriteria problems, such as design, identification, design of controlled systems, operational development of prototypes, analysis of large-scale systems, and multicriteria analysis from observational data.

REFERENCES

1. I.M. Sobol' and R.B. Statnikov, *Selecting Optimal Parameters in Multicriteria Problems*, (in Russian), Nauka, Moscow, (1981).
2. R.B. Statnikov and J.B. Matusov, *Multicriteria Optimization and Engineering*, Chapman & Hall, New York, (1995).
3. R.B. Statnikov and J.B. Matusov, *Multicriteria Analysis in Engineering. Using the PSI Method with MOVI 1.0*, Kluwer Academic, Dordrecht, (2002).
4. R.B. Statnikov, *Multicriteria Design. Optimization and Identification*, Kluwer Academic, Dordrecht, (1999).
5. R.B. Statnikov and J.B. Matusov, Use of P-nets for the approximation of the Edgeworth-Pareto set in multicriteria optimization, *Journal of Optimization Theory and Application* **91** (3), 543–560, (1996).
6. P.C. Dyer, P.C. Fishburn, R.E. Steuer, J. Wallenius and S. Zionts, Multiple-criteria decision making, multiattribute utility theory: The next ten years, *Management Science* **38** (5), 645–654, (1992).
7. W. Stadler and J.P. Dauer, Multicriteria optimization in engineering: A tutorial and survey. Structural optimization: Status and promise, In *Progress in Aeronautics and Astronautics*, Vol. 150, (Edited by M.P. Kamat), pp. 209–249, American Institute of Aeronautics and Astronautics, Washington, DC, (1992).
8. V. Dobrokhodov, R. Statnikov, A. Statnikov and I. Yanushkevich, Modeling and simulation framework for multiple objective identification of a controllable descending system, In *Proceedings of International Conference on Adaptive Modelling and Simulation (ADMOS-2003)*, Goteborg, Sweden, 29 September–1 October 2003.
9. E. Lieberman, *Multi-Objective Programming in the USSR*, Academic Press, New York, (1991).
10. M.G. Parsons and R.L. Scott, Formulation of multicriterion design optimization problems for solution with scalar numerical optimization methods, *Journal of Ship Research* **48** (1), 61–76, (2004).

11. M. Gobbi, G. Mastinu, D. Catelani, L. Guglielmetto and M. Bocchi, Multi-objective optional design of road vehicle sub-systems by means of global approximation, In *Proceedings of the 15^{th} European ADAMS Users' Conference*, (2000).
12. A. Bordetsky, B. Peltsverger, S. Peltsverger and R. Statnikov, Multicriteria approach in configuration of energy efficient sensor networks, In *43^{rd} Annual ACM Southeast Conference ACMSE 2005*, Vol. 2, pp. 28–30, Kennesaw, GA, March 18–20, 2005.
13. I. Yanushkevich, R. Statnikov, A. Statnikov and J. Matusov, *MOVI 1.3 Software Package User's Manual*, Certificate of Registration, United States Copyright Office, The Library of Congress, (2004).
14. R.E. Steuer and M. Sun, The parameter space investigation method of multiple objective nonlinear programming: A computational investigation, *Operations Research* **43** (4), 641–648, (1996).
15. J.H. Halton, On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals, *Numerische Mathematik* **2**, 84–90, (1960).
16. J.M. Hammersley, Monte Carlo methods for solving multivariable problems, *Ann. New York Acad. Sci.* **86**, 844–874, (1960).
17. E. Hlawka, Gleichverteilung auf Produkten von Sphären, *J. Reine Angew. Math.* **30**, 1–30, (1982).
18. H. Faure, Discrépance de suítes associés a une systéme de numération (en dimension s), *Acta Arithmetica* **41**, 337–351, (1982).
19. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, J. Wiley, New York, (1974).
20. H. Niederreiter, Statistical independence properties of pseudorandom vectors produced by matrix generators, *J. Comput. Appl. Math.* **31**, 139–151, (1990).
21. R. Statnikov, A. Bordetsky and A. Statnikov, Multicriteria analysis of real-life engineering optimization problems: Statement and solution, In *Proceedings of the 4^{th} World Congress of Nonlinear Analysts (WCNA)*, Orlando, FL, June 30–July 7, 2004.
22. V. Vapnik, *Statistical Learning Theory*, J. Wiley, New York, (1998).
23. K. Ali Anil, Multicriteria analysis in naval ship design, Master's Thesis, Naval Postgraduate School, Monterey, CA, `http://theses.nps.navy.mil/05Mar_Anil.pdf`, (2005).
24. P.D. Wasserman, *Advanced Methods in Neural Computing*, Van Nostrand Reinhold, New York, (1993).
25. R. Collobert *et al.*, SVMTorch: Support vector machines for large-scale regression problems, *Journal of Machine Learning Research* 1 (February), 143–160, (2001).
26. S. Chatterjee and A.S. Hadi, Influential observations, high leverage points, and outliers in linear regression, *Statistical Science* 1 (3), 379–416, (1986).